MATH 428/828 Notes on Shadow Price and Duality (M428 students not responsible to text in blue)

Consider the primal LP problem of maximizing $z = c_1x_1 + \cdots + c_nx_n$ subject to $a_{i1}x_1 + \cdots + a_{in}x_n \le b_i$ with $b_i \ge 0$ for $i = 1, 2, \ldots, m$ and $x_j \ge 0$ for $j = 1, 2, \ldots, n$. Recall that the simplex method is to maximize the z value from all solutions, $(z, x_1, \ldots, x_{n+m})$, to the augmented linear system of equations: $z - c_1x_1 - \cdots - c_nx_n = 0$, $a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+i} = b_i$ for $1 \le i \le m, x_j \ge 0, 1 \le j \le n+m$ with $x_{n+i} \ge 0$ being the slack variables. There are infinitely many feasible solutions simply because there are more equations than the number of variables, but the optimal solution with maximal value of z usually is only one.

Recall also that the simplex method is to use elementary row reductions to find a feasible echelon form of the augmented system of equations at each step so that a basic feasible solution is obtained by setting all the nonbasic variables zeros and that the corresponding basic feasible solution is to have an improved z-value. Denote the last solution form (the same as in any intermediate step) as

$$\begin{array}{l} z + \bar{c}_1x_1 + \cdots + \bar{c}_{n+m}x_{n+m} = b_0^* = \sum_{i=1}^m y_i^* b_i \\ \quad \bar{a}_{11}x_1 + \cdots + \bar{a}_{1\ (n+m)}x_{n+m} = b_1^* \\ \quad\quad \vdots \\ \quad \bar{a}_{m1}x_1 + \cdots + \bar{a}_{m\ (n+m)}x_{n+m} = b_m^* \\ x_i = 0 \text{ for } n \text{ many non-basic variables and} \\ x_j \ge 0 \text{ for } m \text{ many basic variables.} \end{array}$$

This form is a **feasible echelon form** in the following sense: 1) there are m many integers j with $1 \le j \le n+m$ so that variable x_j is a basic variable because there is an integer $1 \le i_j \le m$ so that the coefficient of x_j from the i_jth constraint equation is $\bar{a}_{i_j j} = 1$, and all other coefficients of x_j are zeros: $\bar{a}_{ij} = 0, i \ne i_j$ and $\bar{c}_j = 0$. That is, the corresponding jth column is a pivot column and the corresponding entry $\bar{a}_{i_j j} = 1$ is the pivot-1 entry; 2) $b_i^* \ge 0$ for all $1 \le i \le m$, satisfying the so-called **Feasibility Test**.

It is the final echelon form if $\bar{c}_i \ge 0$ for all i, namely satisfying the **Optimality Test** and $b_i^* \ge 0$ for all $1 \le i \le m$. The corresponding basic feasible solution is exactly $x_j = b_{i_j}^*$ for all basic variables and $x_i = 0$ for all non-basic variables x_i, with the corresponding z-value at the basic feasible point as $z = b_0^*$.

It is very important to note that b_0^* is a linear combination of the original constraints right hand b_i because of the following reasons. In fact, the coefficient y_i^* of the resource parameter b_i is the result of the elementary row reduction for the simplex method when a y_i^* multiple of the ith constraint row is added to the 0th row. This maybe resulted from one replacement row operation (a constant multiple of a row is added to another row), but more likely it is resulted from a few such operations, and the value y_i^* is only the net or aggregated total effect of such operations involving the ith constraint equation.

More importantly, because the $(n+i)$th slack variable x_{n+i} has the same constant coefficient, 1, as parameter b_i does to start the simplex method, they both have the same coefficient for the z-equation after every row operation, and especially, at the final solution step, both coefficient must be equal: $\bar{c}_{n+i} = y_i^*$.

By definition, the **shadow price** of the ith resource constraint is the rate of change of the optimal solution with respect to the ith resource parameter b_i, that is, $\frac{\partial b_0^*}{\partial b_i}$, which by the z-equation of the last solution form we find

$$\frac{\partial b_0^*}{\partial b_i} = y_i^* = \bar{c}_{n+i}$$

the coefficient of the ith slack variable x_{n+i}. One important practical interpretation is, the shadow price represents the rate of change in the optimal value respect to change in the corresponding resource.

In matrix notation, the primal LP problem is $\max z = c^T x$ sub.to $Ax \le b$, componentwise, with $A = A_{m\times n}$, $c = [c_1, \ldots, c_n]^T, x = [x_1, \ldots, x_n]^T, b = [b_1, \ldots, b_m]^T$ being column vectors. Let $x_s = [x_{n+1}, \ldots, x_{n+m}]^T$ denote the slack variable, $X = \begin{bmatrix} x \\ x_s \end{bmatrix}$, and $\mathbf{0} = [0, 0, \ldots, 0]^T$ the zero-vector. Then the augmented LP form is: $z - c^T x + \mathbf{0}^T x_s = 0, Ax + x_s = b, X \ge \mathbf{0}$ and the problem is to find a feasible solution so that the z value of the solution is the largest. Here for two vectors u, v, $u \le v$ means the inequality holds componentwise. Since y_i^* are the scalar multiples used by the row operations to get the z-equation, at the last step of the simplex method we must have $z - c^T x + {y^*}^T Ax + {y^*}^T x_s = {y^*}^T b$, and equivalently $z = (c^T - {y^*}^T A)x - {y^*}^T x_s + {y^*}^T b := -\bar{c}X + {y^*}^T b$. Notice that this form $z = (c^T - y^T A)x - y^T x_s + y^T b$ holds at any step of row operations when using the components of y as the constant multiples of the corresponding constraint rows and then adding those multiples to the objective row $z - c^T x + \mathbf{0}^T x_s = 0$. When evaluate at the corresponding optimal basic feasible solution X, the z-value of the optimal solution is $z = {y^*}^T b$ for which we must have $c^T - {y^*}^T A \le \mathbf{0}^T$ and $y^* \ge \mathbf{0}$, equivalently, ${y^*}^T A \ge c^T$ or $A^T y^* \ge c, y^* \ge \mathbf{0}$. That is, y^* is a feasible point for this LP problem: $\min w = b^T y$ sub.t. $A^T y \ge c, y \ge \mathbf{0}$, which is called the *dual* LP problem of the primal LP problem. In fact, we have the following duality result, assuming the fact that the LP problems have a solution iff the simplex algorithm will find it in finite steps.

Theorem 1. *The shadow prince y^* for the primal LP problem* $\max z = c^T x$ *sub.to* $Ax \le b, x \ge \mathbf{0}$ *is a solution to the dual LP problem* $\min w = b^T y$ *sub.to* $A^T y \ge c, y \ge \mathbf{0}$.

Proof. The discussion preceded the theorem shows y^* is a feasible point for the dual problem. The only part remains to prove is that the shadow price solves the dual problem. Suppose not, then there is a feasible point $\bar{y}$ for the dual problem so that $A^T \bar{y} \ge c \Leftrightarrow \bar{y}^T A \ge c^T$ and $\bar{y} \ge \mathbf{0}$ but with the dual optimal value $w = b^T \bar{y}$ that is strictly smaller than the feasible value $b^T y^*$, i.e. $b^T \bar{y} < b^T y^*$. We only need to show this inequality is false. To do so, we go back to the primal LP problem. First, instead of using the same operations on the z-equation by the shadow price vector y^*, we use instead the components of $\bar{y}$ for the constant multiples of the corresponding constraint row equations and then add these multiples to the objective row to get the equivalent z-equation: $z = (c^T - \bar{y}^T A)x - \bar{y}^T x_s + \bar{y}^T b$. Next, we use the same row operations on the constraint equations of the primal problem to get the same basic feasible point X since the z-equation is never used in row operations for the constraint equations. Since the row operations do not change the solutions to the equality LP problem, in particular, not the value of the z variable, the evaluation of the new z equation at the same optimal basic feasible point X must produce the same optimal value $z = (c^T - \bar{y}^T A)x - \bar{y}^T x_s + \bar{y}^T b = {y^*}^T b$ with x and x_s consisting of the basic feasible solution X. We now show this is impossible under the assumption that $\bar{y}^T b < {y^*}^T b$. Because $(c^T - \bar{y}^T A) \le \mathbf{0}^T$ and $-\bar{y}^T \le \mathbf{0}$ by the feasibility condition for the dual problem and $x \ge \mathbf{0}, x_s \ge \mathbf{0}$, we surely will have $(c^T - \bar{y}^T A)x - \bar{y}^T x_s \le 0$ and hence ${y^*}^T b \le \bar{y}^T b$. This contradicts the assumption that $\bar{y}^T b < {y^*}^T b$. □

Notice that, the primal and the dual problem have the same optimal value ${y^*}^T b$. As an exercise, prove the dual of the dual problem is the primal problem. That is, the primal and dual problems are the dual problem of each other. Also, prove that the solution x^* to the primal problem is the shadow price for the dual problem, that is $c^T x^* = b^T y^*$.