# Proof of Perron-Frobenius Theorem

Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ with $y^T$ denoting exclusively the transpose of vector $y$. Let $\|x\| = \max_i\{|x_i|\}$ be the norm. Then the induced operator norm for matrix $A = [a_{ij}]$ is $\|A\| = \max_i\{\sum_j |a_{ij}|\}$.

Consider a Markov's chain on $n$ states with transition probabilities $p_{ij} = \Pr(X_{k+1} = i|X_k = j)$, independent of $k$, and $P = [p_{ij}]$ the transition matrix. Then $\sum_{i=1}^n p_{ij} = 1$ for all $j$. Let $p_{ij}^{(t)} = \Pr(X_{k+t} = i|X_k = j)$ and $P^{(t)} = [p_{ij}^{(t)}]$ be the $t$-step transition probability matrix. Then we have $p_{ij}^{(t)} = \sum_\ell p_{i\ell}^{(t-1)} p_{\ell j}$ for all $i, j$. In matrix, $P^{(t)} = P^{(t-1)}P = \cdots = P^t$ which is the $t$-step transition matrix. If $q = (q_1, \ldots, q_n)^T$ is a probability distribution for the Markovian states at a given iterate with $q_i \geq 0, \sum q_i = 1$, then $Pq$ is again a probability distribution for the states at the next iterate. A probability distribution $w$ is said to be a steady state distribution if it is invariant under the transition, i.e. $Pw = w$. Such a distribution must be an eigenvector of $P$ and $\lambda = 1$ must be the corresponding eigenvalue. The existence as well as the uniqueness of the steady state distribution is guaranteed for a class of Markovian chains by the following theorem due to Perron and Frobenius.

**Theorem 1.** *Let $P = [p_{ij}]$ be a probability transition matrix, i.e. $p_{ij} \geq 0$ and $\sum_{i=1}^n p_{ij} = 1$ for every $j = 1, 2, ..., n$. Assume $P$ is irreducible and transitive in the sense that there is a $t \geq 1$ so that $p_{ij}^{(t)} > 0$ for all $i, j$. Then 1 is a simple eigenvalue of $P$ and all other eigenvalues $\lambda$ satisfy $|\lambda| < 1$. Moreover, the unique eigenvector can be chosen to be a probability vector $w$ and it satisfies $\lim_{t\to\infty} P^t = [w, w, \ldots, w]$. Furthermore, for any probability vector $q$ we have $P^t q \to w$ as $t \to \infty$.*

*Proof.* We first prove a claim that $\lim_{t\to\infty} p_{ij}^{(t)}$ exist for all $i, j$ and the limit is independent of $j$, $\lim_{t\to\infty} p_{ij}^{(t)} = w_i$.

Because $P = [p_{ij}]$ (is irreducible and transitive) has non-zero entries, we have

$$\delta = \min_{ij} p_{ij} > 0.$$

Consider the equation of the $ij$th entry of $P^{t+1} = [p_{ij}^{(t+1)}] = P^t P$,

$$p_{ij}^{(t+1)} = \sum_k p_{ik}^{(t)} p_{kj}.$$

Let

$$0 < m_i^{(t)} := \min_j p_{ij}^{(t)} \leq \max_j p_{ij}^{(t)} := M_i^{(t)} < 1.$$

Then, we have

$$m_i^{(t+1)} = \min_j \sum_k p_{ik}^{(t)} p_{kj} \geq m_i^{(t)} \sum_k p_{kj} = m_i^{(t)}.$$

i.e., the sequence $\{m_i^{(1)}, m_i^{(2)}, \dots\}$ is non-decreasing. Similarly, the upper bound sequence $\{M_i^{(1)}, M_i^{(2)}, \dots\}$ is non-increasing. As a result, both limits $\lim_{t\to\infty} m_i^{(t)} = m_i \le M_i = \lim_{t\to\infty} M_i^{(t)}$ exist. We now prove they are equal $m_i = M_i$.

To this end, we consider the difference $M_i^{(t+1)} - m_i^{(t+1)}$:

$$\begin{aligned} M_i^{(t+1)} - m_i^{(t+1)} &= \max_j \textstyle\sum_k p_{ik}^{(t)} p_{kj} - \min_\ell \sum_k p_{ik}^{(t)} p_{k\ell} \\ &= \max_{j,\ell} \textstyle\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell}) \\ &= \max_{j,\ell} [\textstyle\sum_k^+ p_{ik}^{(t)} (p_{kj} - p_{k\ell}) + \sum_k^- p_{ik}^{(t)} (p_{kj} - p_{k\ell})] \\ &\le \max_{j,\ell} [M_i^{(t)} \textstyle\sum_k^+ (p_{kj} - p_{k\ell}) + m_i^{(t)} \sum_k^- (p_{kj} - p_{k\ell})] \end{aligned} \tag{1}$$

where $\sum_k^+ p_{ik}^{(t)} (p_{kj} - p_{k\ell})$ means the summation of all non-negative terms $p_{kj} - p_{k\ell} \ge 0$ and similarly $\sum_k^- p_{ik}^{(t)} (p_{kj} - p_{k\ell})$ means the summation of all negative terms $p_{kj} - p_{k\ell} < 0$.

It is critical to notice the following unexpected equality:

$$\begin{aligned} \textstyle\sum_k^- (p_{kj} - p_{k\ell}) &= \textstyle\sum_k^- p_{kj} - \sum_k^- p_{k\ell} \\ &= 1 - \textstyle\sum_k^+ p_{kj} - (1 - \sum_k^+ p_{k\ell}) \\ &= \textstyle\sum_k^+ (p_{k\ell} - p_{kj}) \\ &= -\textstyle\sum_k^+ (p_{kj} - p_{k\ell}). \end{aligned}$$

Hence, the inequality (1) becomes

$$M_i^{(t+1)} - m_i^{(t+1)} \le (M_i^{(t)} - m_i^{(t)}) \max_{j,\ell} \sum_k{}^+ (p_{kj} - p_{k\ell}).$$

If $\max_{j,\ell} \sum_k^+ (p_{kj} - p_{k\ell}) = 0$, which is independent of all $t$, it is done that $M_i^{(t)} = m_i^{(t)}$. Otherwise, for the pair $j, \ell$ that gives the maximum let $r$ be the number of terms in $k$ for which $p_{kj} - p_{k\ell} > 0$, and $s$ be the number of terms for which $p_{kj} - p_{k\ell} < 0$. Then $r \ge 1$, and $\tilde{n} := r + s \ge 1$ as well as $\tilde{n} \le n$. More importantly

$$\begin{aligned} \textstyle\sum_k^+ (p_{kj} - p_{k\ell}) &= \textstyle\sum_k^+ p_{kj} - \sum_k^+ p_{k\ell} \\ &= 1 - \textstyle\sum_k^- p_{kj} - \sum_k^+ p_{k\ell} \\ &\le 1 - s\delta - r\delta = 1 - \tilde{n}\delta \\ &\le 1 - \delta < 1. \end{aligned}$$

The estimate for the difference $M_i^{(t+1)} - m_i^{(t+1)}$ at last reduces to

$$M_i^{(t+1)} - m_i^{(t+1)} \le (1-\delta)(M_i^{(t)} - m_i^{(t)}) \le (1-\delta)^t (M_i^{(1)} - m_i^{(1)}) \to 0,$$

as $t \to \infty$, showing $M_i = m_i := w_i$. As a consequence to the inequality $m_i^{(t)} \le p_{ij}^{(t)} \le M_i^{(t)}$, we have $\lim_{t\to\infty} p_{ij}^{(t)} = w_i$ for all $j$. In matrix notation, $\lim_{t\to\infty} P^t = [w, w, \dots, w] := W$, a matrix of equal column vectors.

Next, assume there is a $k \geq 1$ so that only $p_{ij}^{(k)} > 0$ for all $i, j$. Then the result above implies $P^{kt} \to W$ as $t \to \infty$. We need to show that $P^t \to W$ as $t \to \infty$ as well. This is left as an exercise.

Next, we show that $\lambda = 1$ is an eigenvalue with eigenvector $w$. In fact from the definition of $w$ above $\lim_{t\to\infty} P^t = [w, w, \ldots, w]$ and thus $[w, w, \ldots, w] = \lim_{t\to\infty} P^t = P \lim_{t\to\infty} P^{t-1} = P[w, w, \ldots, w] = [Pw, Pw, \ldots, Pw]$ showing $Pw = w$.

To show that the eigenvalue $\lambda = 1$ is simple, two cases are considered. First, let $x \neq 0$ be an eigenvector $Px = x$ of the eigenvalue 1. Apply $P$ to the identity repeatedly to have $P^t x = x$. In limit, $\lim_{t\to\infty} P^t x = Wx = [w, w, \ldots, w]x = wx_1 + wx_2 + \cdots + wx_n = (\sum x_j)w = x$. Denote it by $\bar{x} = \sum_j x_j$. Then $x_i = \bar{x}w_i$ for all $i$. Because $x \neq 0$, we must have $\bar{x} = \sum_j x_j \neq 0$, and that $x = \bar{x}w$ for some constant $\bar{x} \neq 0$, showing that the eigenvector $w$ is unique up to a constant multiple. Second, let $x \neq 0$ be a generalized eigenvector of $\lambda = 1$. Then there is a constant $c \neq 0$ so that $Px = x + cw$ which implies $P^t x = x + ctw$. Since $\lim_{t\to\infty} P^t x = [w, w, \ldots, w]x$ exists on the left, we must have $c = 0$ on the right, a contradiction. Together we can conclude that the dimension of the generalized eigenspace for $\lambda = 1$ is 1, i.e., the eigenvalue $1$ is simple. In addition, for any probability vector $q$, the result above shows $\lim_{t\to\infty} P^t q = Wq = \sum_j q_j w = w$ as $Wx = \sum_j x_j w$ always holds.

Next, let $\lambda$ be an eigenvalue of $P$. Then it is also an eigenvalue for the transpose $P^T$. Let $x$ be an eigenvector of $\lambda$ of $P^T$. Then $P^T x = \lambda x$ and $\|\lambda x\| = |\lambda|\|x\| = \|P^T x\| \leq \|P^T\|\|x\|$. Since $\|P^T\| = 1$ because $\sum_{i=1}^n p_{ij} = 1$ we have $|\lambda| \leq 1$.

Next, let $x$ be an eigenvector of an eigenvalue $\lambda$. Then we have $\lim_{t\to\infty} P^t x = Wx = (\sum x_j)w$ on one hand and $\lim_{t\to\infty} P^t x = \lim_{t\to\infty} \lambda^t x$ on the other hand. So either $|\lambda| < 1$ in which case $\lim_{t\to\infty} \lambda^t x = 0$ and then $\sum x_j = 0$, or $|\lambda| = 1$ in which case $\lambda = e^{i\theta}$ for some $\theta$ and the limit $\lim_{t\to\infty} \lambda^t = \lim_{t\to\infty} e^{i\theta t}$ exists since $\lim_{t\to\infty} e^{i\theta t} x = \lim_{t\to\infty} \lambda^t x = \lim_{t\to\infty} P^t x = (\sum x_j)w$. The latter case holds if and only if $\sum x_j \neq 0$ and $\theta = 0$, i.e., $\lambda = 1$. This shows that all eigenvalues that is not $\lambda = 1$ are inside the unit circle and the corresponding eigenspace is $\{x : \sum_j x_j = 0\}$, which $n - 1$ dimensional. □

References: Bellman(1997); Berman & Plemmons(1994); Frobenius(1908, 1912); Lancaster & Tismenetsky(1985); Marcus & Minc(1984); Perron(1907); Petersen(1983); Seneta(1973).

Ethier and Kurtz, Markov Processes – Characterization and Convergence.