Math 314 Summer '12 Exam 2

Name: ____________________ **Score:** ____________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (10 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) T or F? λ is an eigenvalue of A if and only if $\text{null}(A - \lambda I)$ has a nonzero vector.

   (b) T or F? If an invertible matrix A is similar to matrix B then $\det(B) = \det(A^{-1})$.

   (c) T or F? If W is the null space of a matrix A then the orthogonal complement $W^{\perp}$ is the row space of A.

   (d) T or F? The dimension of the space of polynomials of degree n, $\mathbb{P}_n$ is n.

   (e) T or F? If $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is a change-of-coordinate matrix, then 1 must be one of its eigenvalues.

   (f) T or F? If $\mathbf{v_1}$ and $\mathbf{v_2}$ are orthogonal, then $A\mathbf{v_1}$ and $A\mathbf{v_2}$ must be orthogonal for any invertible matrix A.

   (g) T or F? The rank of any $n \times n$ orthogonal matrix is always 0.

2. (10 pts) The following matrix A and B are row equivalent:

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for the row space of A.

   (b) Find a basis for the row space of A that are row vectors of A. Justify your answer.

3. (15 pts) Let $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ be a basis of $\mathbb{P}_2$ and $\mathcal{C} = \{1, t, t^2\}$ be its standard basis.

   (a) Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

   (b) Write t^2 as a linear combination of the polynomials in $\mathcal{B}$ and the coordinate $[t^2]_{\mathcal{B}}$.

4. (5 pts) Let $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$ be a basis of $\mathcal{M}_{2\times2}$, and $[A]_{\mathcal{B}} = \begin{bmatrix}0\\1\\2\\3\end{bmatrix}$. Find A.

5. (10 pts) Let $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Find a vector $\mathbf{w} \in W = \text{span}\{\mathbf{v}_1\}$ and $\mathbf{w}^\perp \in W^\perp$ so that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. Show work to justify your answer.

6. (15 pts) The eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ are 2 and 5. Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$, if possible.

7. (10 pts) (a) Find the characteristic equation for the matrix $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(b) Find all eigenvalues of A.

8. (15 pts) (a) Verify that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ with $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

(b) Find the coordinate $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$.

(c) Find the distance between $\mathbf{v}$ and $\mathbf{v}_3$.

9. (5 pts) Let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors of two distinct eigenvalues λ_1, λ_2, respectively, for a symmetric matrix A. Prove that $\mathbf{v}_1, \mathbf{v}_2$ are perpendicular.

10. (5 pts) Let A be an orthogonal matrix (i.e. $A^{-1} = A^T$). Prove that its row vectors form an orthonormal set.

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2 Bonus Points: The word *algebra* comes from (a) Latin, (b) Greek, (c) Arabic, or (d) Hindu.