Math 314 Summer '12 Final Exam

Name: ______________________ **Score:** ______________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (22 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) **T** or **F**? The rank of a matrix is always greater than zero.

   (b) **T** or **F**? A symmetric matrix can have a zero singular value.

   (c) **T** or **F**? $\mathbf{0}$ can be an eigenvector of a symmetric matrix.

   (d) **T** or **F**? If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of linearly dependent vectors, then one of the vectors must be a scalar multiple of another one of the vectors.

   (e) **T** or **F**? Let A by an $n \times n$ matrix. If the rows of A span $\mathbb{R}^n$, then the columns of A must be linearly independent.

   (f) **T** or **F**? Elementary row operations preserve the column space of a matrix.

   (g) **T** or **F**? The row space of a matrix is the orthogonal to its column space.

   (h) **T** or **F**? $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} [1/\sqrt{2}, 1/\sqrt{2}] - \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} [-1/\sqrt{2}, 1/\sqrt{2}]$ is the spectral decomposition of a symmetric matrix.

2. (10 pts) Find all real number k such that the product of the matrixes $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & k \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ k & 1 & 3 \\ 0 & 2 & 0 \end{bmatrix}$ is invertible.

3. (35 pts) The following matrices are row equivalent

$$A = \begin{bmatrix} 10 & 0 & 20 & 3 & -4 \\ -3 & 1 & -7 & -1 & 4 \\ 3 & 0 & 6 & 1 & -1 \\ 3 & 1 & 5 & 1 & 2 \\ 2 & 0 & 4 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the reduced row echelon form, $rref(A)$, of A.

(b) Find the rank and the nullity of A.

(c) Find a basis for the column space of A.

(d) Write the fifth column of A as a linear combination of the other columns.

(e) Find a basis for the subspace W that is orthogonal to the nullspace of A.

4. (18 pts) (a) Find the inverse matrix A^{-1} of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$.

(b) Write A as a product of elementary matrices.

5. (15 pts) Let $W = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \}$.

(a) Find a base for $W^{\perp}$

(b) Decompose $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ as $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ with $\mathbf{w}$ from W and $\mathbf{w}^{\perp}$ from $W^{\perp}$.

6. (10 pts) Let T be a linear transformation that projects any vector $\mathbf{x}$ in $\mathbb{R}^3$ to the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find its matrix of transformation $[T]$.

7. (18 pts) You are given the fact that the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 2 & -2 \\ -2 & 0 & 4 \end{bmatrix}$ are 2 and 1.

(a) Find a basis for the eigenspace E_λ for each eigenvalue.

(b) If possible, diagonalize the matrix A by finding P and D to factorize A as $A = PDP^{-1}$.

8. (15 pts) Let $\mathcal{B} = \{2+t, 5+3t\}$ and $\mathcal{C} = \{1, t\}$ be two bases of $\mathbb{P}_2$.

(a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$

(b) If $\mathbf{p}(t) = 1 + 2t$, find both $[\mathbf{p}]_{\mathcal{C}}$ and $[\mathbf{p}]_{\mathcal{B}}$.

9. (15 pts) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

10. (10 pts) Find the singular values of $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$.

11. (15 pts) You are given the fact the singular values of the matrix $A = \begin{bmatrix} 63 & 0 & 16 \\ 16 & 0 & 63 \end{bmatrix}$ are 79 and 47. Find the singular value decomposition of A in exact values.

12. (12 pts) Prove that the eigenvalues of $A^T A$ for any $m \times n$ matrix A must be nonnegative.

13. (5 pts) The *length* of a matrix $A = [a_{ij}]_{m \times n}$ is defined as $\|A\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$. Let $A = \mathbf{u}\mathbf{v}^T$ with $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. Show that if $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 1$ then $\|A\| = 1$.

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2 Bonus Points: The number of times that homework are collected are (a) 16, (b) 18, (c) 20, (d) 22.