Solu. Key to Test 2 Spring 2014 Math 314

1 (15 pt) (a) T (b) F (c) T (d) F (e) F (f) F

2 (15 pts) (a) $-1, 0, 1$ double (b) $\lambda = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, (c) yes, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2$ in $E_{\lambda=1}$.

3 (15 pts). (a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (b) $P_{C \leftarrow B} = [[b_1]_C, [b_2]_C]$

$$[C \mid B] = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(c) $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. $[x]_C = P_{C \leftarrow B} [x]_B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(check. $-\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. ✓)

4 (15 pts) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ column 1, 2, ~~3~~ 4 linearly indep.

$\Rightarrow \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ basis for span$\{v_1, v_2, v_3, v_4\}$.

5 (15 pts) • $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) = 0$, $\lambda = 1, 2$.

• for $\lambda = 1$. $(A - \lambda I)\vec{v} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $x_1 = 0$, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

• for $\lambda = 2$ $(A - \lambda I)\vec{v} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $x_1 = x_2$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$P = [v_1\ v_2] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = -\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

6 (15 pts) (a) $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -\begin{vmatrix} 1 & -2-\lambda & 1 \\ -2-\lambda & 1 & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -\begin{vmatrix} 1 & -2-\lambda & 1 \\ 0 & 1-(2+\lambda)^2 & 3+\lambda \\ 0 & 3+\lambda & -3-\lambda \end{vmatrix}$

$= -\begin{vmatrix} 1-(2+\lambda)^2 & 3+\lambda \\ 3+\lambda & -3-\lambda \end{vmatrix} = -\begin{vmatrix} 1-(2+\lambda)^2 & 3+\lambda \\ 3+\lambda+(1-(2+\lambda)^2) & 0 \end{vmatrix} = (3+\lambda)(3+\lambda+(3+\lambda)(-1-\lambda))$

$= -(3+\lambda)^2 \lambda = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = -3$, $\lambda_3 = -3$.

(b) $A\vec{v} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. no.

7 (15 pts) Solve $(A - \lambda I)\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vec{x} = 0$. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\Rightarrow x_1 + x_2 + x_3 = 0$. $x_1 = -x_2 - x_3$, $\Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$.

$\Rightarrow E_{\lambda=-3} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2 Bonus pts. : F