Math 314 Fall '10 Exam 2

Name: ______________________ **Score:** ______________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (20 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) T or F? A matrix A is invertible if and only if $\det A = 0$.

   (b) T or F? λ is an eigenvalue of A if and only if $\text{null}(A - \lambda I)$ has a nonzero vector.

   (c) T or F? If $\det(A) = 2, \det(B^{-1}) = 2$ then $\det(AB) = 4$.

   (d) T or F? The row space $\text{row}(A)$ of a matrix A is always orthogonal to its column space $\text{col}(A)$.

   (e) T or F? If an invertible matrix A is similar to matrix B then $\det(B) = -\det(A^{-1})$.

   (f) T or F? If W is the null space of a matrix A then the orthogonal complement $W^{\perp}$ is spanned by the rows of A.

   (g) T or F? An invertible matrix A is always diagonalizable.

   (h) T or F? A symmetric matrix A is always invertible.

   (i) T or F? An eigenvector of an invertible matrix A is always orthogonal to any eigenvector of a different eigenvalue of A.

   (j) T or F? If the characteristic equation of a matrix A is given by $p(\lambda) = (\lambda - 2)^3 q(\lambda)$ for some polynomial $q(x)$ then the eigenspace E_2 must be 3 dimensional.

2. (10 pts) Given the following similarity relation $P^{-1}AP = D$.

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}}^{A} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   (a) What are the eigenvalues of A and their algebraic multiplicities?

   (b) Give a basis for the eigenspace of the smallest eigenvalue.

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3. (15 pts) Let Let $A = \begin{bmatrix} 5 & -6 & 4 & 4 \\ 8 & -13 & 8 & 12 \\ 0 & -4 & 1 & 8 \\ 4 & -8 & 4 & 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$.

(a) Use a combination of row and column expansion and elementary row reduction to find the determinant of A.

(b) Verify that $\mathbf{v}$ is an eigenvector. What is the corresponding eigenvalue?

4. (10 pts) Diagonalize matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, i.e., find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. (You don't need to find P^{-1}.)

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5. (15 pts) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

6. (10 pts) You are given the fact that $\lambda = 1$ is an eigenvalue of $A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & -1 & -2 \\ -2 & 4 & 5 \end{bmatrix}$. Find a basis for its eigenspace E_1.

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7. (15 pts) Let $A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & 2 \end{bmatrix}$.

(a) Find the characteristic equation of A.

(b) You are given the fact that vectors $\begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors of A. Find an orthogonal diagonalization of A, i.e. find an orthogonal matrix Q and a diagonal matrix D so that $A = QDQ^T$.

8. (5 pts) Let W be a subspace of $\mathbb{R}^n$ and $W^\perp$ be its orthogonal complement. Prove that if a vector $\mathbf{v}$ is both from W and $W^\perp$, i.e., $\mathbf{v} \in W \cap W^\perp$, then $\mathbf{v}$ must be the zero vector $\mathbf{v} = \mathbf{0}$.

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2 Bonus Points: True of False: The Canadians celebrate the Thanksgiving on the same day as the Americans.

(... *The End*)