NAME: ______________________________

*INSTRUCTIONS: No references, __**no calculators**__, no cellphones. Please **clearly organize your solutions** and emphasize the answers. You must show all your work to receive credit. When unable to solve, you may earn partial credit by outlining which solution strategy you would attempt.*

The exam has a total of 65 points (= 100%).

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1. (1 pt each) Suppose A is $n \times n$ matrix. List four different conditions equivalent to A being invertible

   (i)

   (ii)

   (iii)

   (iv)

2. (6 pts) Suppose $AB\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $\mathbf{x}$ if you know that

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

3. (8 pts) Find the inverse of the given matrix using the Gauss-Jordan method:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4. The following matrices (of dimensions 5×6) are row-equivalent:

$$A = \begin{bmatrix} 0 & 5 & 0 & -2 & 0 & 6 \\ 4 & -2 & 8 & 3 & 4 & -9 \\ 1 & -1 & 2 & -1 & 1 & 3 \\ 2 & -1 & 4 & -1 & 2 & 3 \\ 1 & 7 & 2 & 3 & 1 & -9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the following questions and **briefly** justify your reasoning:

(a) (2 pts) What is the rank of A?

(b) (3 pts) Are the rows of A linearly independent?

(c) (3 pts) Is the 6th (the last) column of A a linear combination of the first five columns?

(d) (2 pts) What is the nullity of A?

5. (10 pts) For the same row-equivalent matrices as in the previous problem:

$$A = \begin{bmatrix} 0 & 5 & 0 & -2 & 0 & 6 \\ 4 & -2 & 8 & 3 & 4 & -9 \\ 1 & -1 & 2 & -1 & 1 & 3 \\ 2 & -1 & 4 & -1 & 2 & 3 \\ 1 & 7 & 2 & 3 & 1 & -9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for the null space of A.

6. (5 pts) Let set S contain all points inside the sphere of radius 1: $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ : \ x^2 + y^2 + z^2 \leq 1 \right\}$

   Is S a subspace of $\mathbb{R}^3$? Either prove or provide a counterexample.

7. (2 pts) Solve for X: $\quad \left(X^T + B\right)^T = A$

8. (3 pts) Suppose $\quad (A - B)^{-1} = C^{-1}B \quad$ (here the matrices $(A - B)$ and C are invertible). Show that B must be invertible too.

9. The following vectors form a basis for $\mathbb{R}^3$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   (a) (5 pts) What are the coordinates of vector $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to this basis?

   (b) (1 pts) Now suppose the coordinates of some vector $\mathbf{w}$ with respect to $\mathcal{B}$ are $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   What is $\mathbf{w}$ in the standard coordinates?

10. (5 pts) Let A, B be two square matrices, and let "0" denote the zero matrix. Either prove or provide a counterexample to the statement:

$$\textit{If } AB = 0 \textit{ and } A \neq 0 \textit{ then } B = 0$$

11. (6 pts) Let A be $m \times n$. Show that every vector in $\text{row}(A)$ is orthogonal to every vector in $\text{null}(A)$.