NAME: ______________________________

*INSTRUCTIONS:* ***Clearly organize*** *your solutions and* ***emphasize the answers****. You must show all your work to receive credit. No references,* ***no electronic devices*** *(short of your wristwatch).*

*You may earn partial credit by outlining which solution strategy you would attempt, and writing down relevant formulas and equations that you remember.*

*This exam has 95 points (=100%) and 10 bonus points (another* $\approx 10.5\%$*).*

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1. (5pts) Let $A$ be an $n \times n$ matrix. Write 5 distinct conditions equivalent to stating that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution for <u>any</u> vector $\mathbf{b}$ in $\mathbb{R}^n$.

(i)

(ii)

(iii)

(iv)

(v)

2. (7 pts) Find $\det \begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & 1 & -2 & -3 \end{bmatrix}$

3. (5 pts) Give an example showing that $(A+B)^{-1} \neq A^{-1}+B^{-1}$ in general for two square **invertible** matrices $A$, $B$ (of the same dimensions).

4. (6 pts) Give a basis for col($A$)

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 5 & -1 & 1 \\ 0 & 5 & -1 & 1 \end{bmatrix}$$

5. (6pts) Find the dimension of the vector space $V$ and give a basis for $V$:

$$V = \{p(x) \quad \text{in} \quad \mathcal{P}_2 : p(1) = 0\}$$

(recall that $\mathcal{P}_2$ denotes the vector space of all polynomials of degree $\leq 2$)

6. Let $A, B$ be $n \times n$ invertible matrices such that

$$A(X+I)B = (AB)^{-1}$$

   (a) (6 pts) Solve for the matrix $X$. Simplify your answer as much as possible.

   (b) (2 pts) Does $X$ have the eigenvalue $\lambda = -1$? Explain.

7. (6pts) Let $A_1, A_2, \ldots, A_k$ be invertible $n \times n$ matrices. Prove or provide a counterexample to the statement: the product $A_1 A_2 \ldots A_k$ must also be invertible.

8. Consider the matrix: $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

(a) (10 pts) Find the eigenvalues of $A$ and state their algebraic multiplicities.

(b) (2 pts) Based on the eigenvalues and their algebraic multiplicities can you tell whether $A$ is diagonalizable? (**briefly justify** your answer)

9. (10pts) A certain $3 \times 3$ matrix $A$ has eigenvalues $5, 6, -1$ and eigenvectors corresponding to these eigenvalues are respectively:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Vector $\mathbf{v}$ written with respect to the basis $\mathcal{B}$ formed by the above eigenvectors (in the same order) has the coordinates

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Find $\mathbf{v}$ and $A^{2008}\mathbf{v}$. Simplify the result where possible.

10. Some $3 \times 3$ matrix $M$ has eigenvalues 1, 0, 2 and corresponding eigenvectors respectively are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) (2pts) Write down a diagonal matrix that is similar to $M$.

(b) (9pts) Find $M$.

11. Matrix $A = \begin{bmatrix} -7 & -3 & 0 & -3 \\ 3 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 3 & 3 & 0 & -1 \end{bmatrix}$ has an eigenvalue $\lambda = -4$.

(a) (6 pts) Find a basis for the corresponding eigenspace $E_{(-4)}$.

(b) (5 pts) Find an **orthonormal** basis for the orthogonal complement $E_{(-4)}^{\perp}$.

12. (8pts) Write down **the second column** of the matrix corresponding to the transformation $T$ which projects every vector in $\mathbb{R}^3$ onto the space $W = \text{span}\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

**Bonus I. (3pts)** Find the null space of the matrix corresponding to the above transformation.

13. (**Bonus II**: 7pts. Very little partial credit).

Let $A$ be an invertible $n \times n$ matrix, and $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ be the solution to the linear system $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b} = [b_1, b_2, \ldots, b_n]^T$. Then *Cramer's rule* gives an explicit formula for each coordinate of $\mathbf{x}$:

$$\boxed{x_i = \frac{\det A_i[\mathbf{b}]}{\det A} \quad \text{for} \quad i = 1, 2, \ldots n}$$

where the matrix $A_i[\mathbf{b}]$ is obtained by replacing the $i$-th column of $A$ with the vector $\mathbf{b}$.

$$\left( \text{For example, if} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad \text{then} \quad A_2[\mathbf{b}] = \begin{bmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{bmatrix} \right)$$

**Prove this result**.

*Hint: How does* $\mathbf{b}$ *relate to the columns of* $A$ *and the numbers* $x_i$*? What do you know about the determinant of a matrix and the determinant of its transpose? How do row operations affect the determinant?*