hwk_w1-1

Ch1.2, #3: Reduce the matrix to reduced echelon form. Circle pivot positions in the final matrix and in the original matrix, and list the pivot columns

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Solu: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \xrightarrow[-6R_1+R_3]{-4R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \xrightarrow[-\frac{1}{5}R_3]{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$\xrightarrow[-2R_2+R_1]{-R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. pivot columns: col 1, col 2, $\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$

#14. Find solutions of the system whose augmented matrix is

$$[A:b] \sim \begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solu: $[A:b] \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Longleftrightarrow \begin{aligned} x_1 \quad + 7x_3 \quad &= -9 \\ x_2 - 6x_3 - 3x_4 &= 2 \\ x_5 &= 0 \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4$$

hwk_w1-2

Ch1.3, #8. Write $\vec{w}, \vec{x}, \vec{y}, \vec{z}$ as linear combinations of $\vec{u}$ and $\vec{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\vec{u}$ and $\vec{v}$?

$\vec{w} = 2\vec{v} - \vec{u}$, $\vec{x} = 2\vec{v} - 2\vec{u}$, $\vec{y} = \frac{3}{2}\vec{v} - 2\vec{u}$, $\vec{z} = 4\vec{v} - 3\vec{u}$. Yes, every vector in $\mathbb{R}^2$ is a linear combination of $\vec{u}$ and $\vec{v}$

#13: Determine if $\vec{b}$ is a linear combination of the vectors formed from the columns of A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

Solu: let $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$. Then $\vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 = \vec{b}$ has a solution in $\vec{x}$ if and only if (iff) $A\vec{x} = \vec{b}$.

Solve it by rref:

$$[A\, b] = \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

which is inconsistent because the last equation is $0x_1 + 0x_2 + 0x_3 = 3$, a contradiction. So $\vec{b}$ is not a linear combination of the column vectors of A. □

hwk_w2-1

Ch1.4, #12: Write the augmented matrix for $A\vec{x} = \vec{b}$. Then solve the system and write the solution as a vector

$$A = \cancel{\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}}, \cancel{\vec{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}} \quad \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

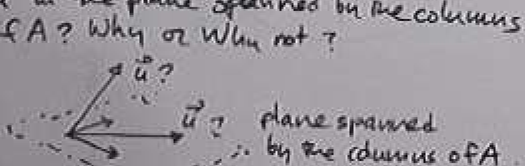
Solu: $[A \vdots \vec{b}] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$

$\xrightarrow{3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{-R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & -2 & -2 \end{bmatrix}$

$\xrightarrow[-\frac{1}{2}R_3]{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & -2/5 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$

#13: Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ in the plane spanned by the columns of A? Why or Why not?



Solu. let $A = [\vec{a}_1, \vec{a}_2]$, Solve $A\vec{x} = \vec{u}$ by rref

$[A;\vec{u}] = \begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$, Yes, $\vec{u} = (\frac{5}{2})\vec{a}_1 + (\frac{3}{2})\vec{a}_2$. □

hwk_w2-2

Ch1.5, #16: Describe the solutions in parametric form, and provide a geometric comparison with the homogeneous solution,

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Solu: $A\vec{x} = \vec{b}$, $[A, \vec{b}] \rightarrow \text{rref} = \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution to $A\vec{x} = \vec{b}$: $\vec{x} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3$

solution to $A\vec{x} = \vec{0}$: $\vec{x} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} x_3$. Them are parallel lines, the former is through the point $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ and the latter is through the origin, both with direction vector $\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

#24: Mark each statement as True or False

a. If $\vec{x}$ is a nontrivial solution of $A\vec{x} = \vec{0}$, then every entry in $\vec{x}$ is nonzero. (False)

b. The equation $\vec{x} = x_2\vec{u} + x_3\vec{v}$ with x_2 and x_3 free (and neither $\vec{u}$ nor $\vec{v}$ a multiple of the other) describes a plane through the origin. (True)

c. The equation $A\vec{x} = \vec{b}$ is homogeneous if the zero vector is a solution. (True)

d. The effect of adding $\vec{p}$ to a vector is to move the vector in a direction parallel to $\vec{p}$. (OK)

e. The solution set of $A\vec{x} = \vec{b}$ is obtained by translating the solution set of $A\vec{x} = \vec{0}$. (OK)

hwk_w2-3

ch 1.6 #7: Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide:

$NaHCO_3 + H_3C_6H_5O_7 \to Na_3C_6H_5O_7 + H_2O + CO_2$

Balance the chemical equation.

Solu: Let x_1, x_2, x_3, x_4, x_5 be the amount of molecules, respectively, for the reactants and products. Then

$$\begin{array}{c} Na \\ H \\ C \\ O \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} x_2 = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} x_5$$

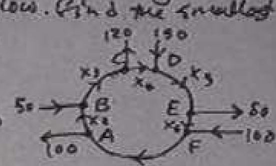
$$\Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix} \vec{x} = \vec{0}, \quad A\vec{x} = \vec{0}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} t$$

for any ~~natural~~ integer/or real number $t \geq 0$.

That is, 3:1:1:3:3 is the stoichiometric ratio for the reaction.

#14: Intersections in England are often constructed as one-way "roundabouts", such as the one shown in the figure. Assume that traffic must travel in the direction shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



Solu:

Junction	In-flow = Outflow
A	$x_1 = x_2 + 100$
B	$x_2 + 50 = x_3$
C	$x_3 = x_4 + 120$
D	$x_4 + 150 = x_5$
E	$x_5 = x_6 + 80$
F	$x_6 + 100 = x_1$

$$A\vec{x} = \vec{b} \Rightarrow [A \vdots \vec{b}] = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$$\Rightarrow \text{rref}([A \vdots b]) = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = x_6 + 100$, $x_2 = x_6$, $x_3 = x_6 + 50$, $x_4 = x_6 - 70$, $x_5 = x_6 + 80$.

$x_6 \geq 0$ free variable.

But since $x_4 = x_6 - 70 \geq 0$, $\Rightarrow x_6 \geq 70$.

The practical solutions are

$x_1 = x_6 + 100$; $x_4 = x_6 - 70$

$x_2 = x_6$; $x_5 = x_6 + 80$

$x_3 = x_6 + 50$; $x_6 \geq 70$.

hwk_w3_1

Ch1.7: #14: Find the value h for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

Solu: $A = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ 3 & 8 & h \end{bmatrix} \xrightarrow[-3R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & 23 & h-3 \end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 23 & h-3 \end{bmatrix} \xrightarrow{-23R_2+R_3} \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h-26 \end{bmatrix}$

For linear dependence, column 3 must not be a pivot column, and $h-26$ must not be a pivot entry, that is

$h - 26 = 0$ and $h = 26$

ch1.7 #42: Use as many columns of A as possible to construct a matrix B with the property that the equation $Bx = 0$ has only the trivial solution. Solve $B\vec{x} = 0$ to verify your work.

$$A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Solution: Use Matlab or Calculator to find rref of A:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Column 1, 2, 4, 6 are pivot columns.

$$\text{So } B = [\vec{a}_1, \vec{a}_2, \vec{a}_4, \vec{a}_6] = \begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$$

Solve $B\vec{x} = 0$, by finding

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

The only trivial solution $\vec{x} = 0$ for $B\vec{x} = 0$

-hwk_w3_2

Ch1.8, #22: Mark each statement True or False. Justify each answer

a. Every matrix transformation is a linear transformation

b. The codomain of the transformation $\vec{x} \mapsto A\vec{x}$ is the set of all linear combinations of the columns of A

c. If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if $\vec{c}$ is in $\mathbb{R}^m$, then a uniqueness question is "Is $\vec{c}$ in the range of T?"

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

e. The superposition principle is a physical description of a linear transformation.

Soln: (a) True. (b) False, $A: \mathbb{R}^n \to \mathbb{R}^m$, $\mathbb{R}^m$ is the codomain, $A\vec{x}$ is the range for all $\vec{x}$ in $\mathbb{R}^n$.

(c) False, a uniqueness question is "Is there a unique $\vec{x}$ so that $T(\vec{x}) = \vec{c}$?"

(d) True since $T(a\vec{x}+b\vec{y}) = aT(\vec{x})+bT(\vec{y})$ by definition.

(e) False, it is a description for the solution of $A\vec{x} = \vec{b}$ for which $\vec{x} = \vec{x}_h + \vec{p}$ where $A\vec{x}_h = 0$ and $A\vec{p} = \vec{b}$ if $A\vec{x} = \vec{b}$ is consistent.

Ch1.8 #38: The matrix

$$A = \begin{bmatrix} -9 & -4 & -9 & 4 \\ 5 & -8 & -7 & 6 \\ 7 & 11 & 16 & -9 \\ 9 & -7 & -4 & 5 \end{bmatrix}$$

determines a linear transformation T. Find all $\vec{x}$ such that $T(\vec{x}) = 0$.

Soln: For $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = 0$, we only need to solve the homogeneous equation $A\vec{x} = \vec{0}$. By rref/ERO, we have

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & -7/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by Calculator.}$$

so $A\vec{x} = 0 \Rightarrow$

$$\begin{aligned} x_1 \quad + 3/4\, x_4 &= 0 \\ x_2 + 5/4\, x_4 &= 0 \\ x_3 - 7/4\, x_4 &= 0 \end{aligned}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -5/4 \\ 7/4 \\ 1 \end{bmatrix} x_4 = \begin{bmatrix} -3 \\ -5 \\ 7 \\ 4 \end{bmatrix} s \quad \text{for any } s.$$

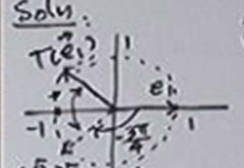
$\Rightarrow$ The null set of T is

$$\text{Null}(T) = \{ \vec{x} : T(\vec{x}) = \vec{0} \} = \text{Span}\left\{ \begin{bmatrix} -3 \\ -5 \\ 7 \\ 4 \end{bmatrix} \right\}.$$

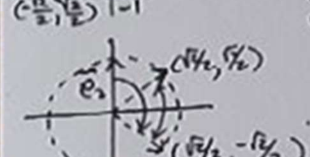
hwk-w3_3

Ch1.9 #7: Assume T is a linear map. Find the standard matrix of T. $T: \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radian and then reflects points through the horizontal axis.

Soln:



$$\vec{e}_1 \xrightarrow{-\frac{3\pi}{4}\text{-rotation}} \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} \xrightarrow{x_1\text{-reflection}} \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = T(\vec{e}_1). \quad T(\vec{e}_1) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.$$



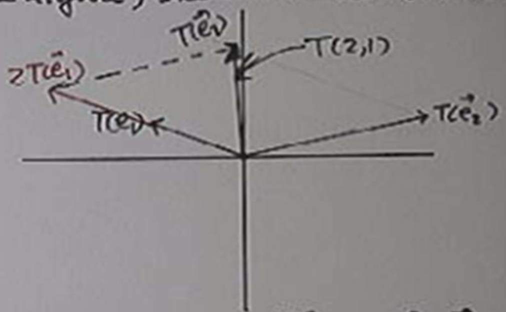
$$\vec{e}_2 \xrightarrow{-\frac{3\pi}{4}\text{ rotation}} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$\xrightarrow{x_1\text{-reflection}} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = T(\vec{e}_2)$$

$\Rightarrow$ Standard matrix $A = [T(\vec{e}_1), T(\vec{e}_2)]$

$$= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}. \text{ And } T(\vec{x}) = A\vec{x} = \frac{\sqrt{2}}{2}\begin{bmatrix} -x_1+x_2 \\ x_1+x_2 \end{bmatrix}$$

#13: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\vec{e}_1)$ and $T(\vec{e}_2)$ are the vectors shown. Using the figure, sketch the vector $T(2,1)$



solution: $T(2,1) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = T(2\vec{e}_1 + \vec{e}_2)$

$= 2T(\vec{e}_1) + T(\vec{e}_2)$.

hwk_w4-1

1) Ch2.1: #11: Let $A=\begin{bmatrix}1&1&1\\1&2&3\\1&4&5\end{bmatrix}$ and $D=\begin{bmatrix}2&0&0\\0&3&0\\0&0&5\end{bmatrix}$

Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3x3 matrix, not the identity matrix or the zero matrix, such that $AB=BA$.

<u>Soln</u>: $AD=[\vec{a}_1,\vec{a}_2,\vec{a}_3]\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}=[\vec{a}_1\lambda_1,\vec{a}_2\lambda_2,\vec{a}_3\lambda_3]$

$DA=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}=\begin{bmatrix}\lambda_1A_1\\\lambda_2A_2\\\lambda_3A_3\end{bmatrix}$.

For any diagonal matrix such as D, for which ~~all~~ nonzero entries appear only on the diagonal, the columns of AD are scalar multiples of the ~~…~~ same columns of A with the scalars from the corresponding diagonal entries of D. Similarly, the rows of DA are scalar multiples of the same rows of A with the scalars from the corresponding diagonal entries of D. ~~Here~~

$AD=\begin{bmatrix}2&3&5\\2&6&15\\2&12&25\end{bmatrix}$, $DA=\begin{bmatrix}2&2&2\\3&6&9\\5&20&25\end{bmatrix}$.

Let $B=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=2I$. Then $AB=A(2I)$
$=2(AI)=2A$. $BA=(2I)A=2(IA)$
$=2A$.

2) Let $A=\begin{bmatrix}1&1&1\\0&2&-2\\2&4&0\end{bmatrix}$.

(a) Find rref(A)

(b) Find the elementary matrix for each of the elementary row operations which are used to obtain rref(A) in (a).

(c) Write rref(A) as a product of elementary matrices and A. Verify the identity by evaluating the multiplications.

<u>Soln</u>. (a) and (b)

$A=\begin{bmatrix}1&1&1\\0&2&-2\\2&4&0\end{bmatrix}\xrightarrow{-2R_1+R_3}\begin{bmatrix}1&1&1\\0&2&-2\\0&2&-2\end{bmatrix}\xrightarrow{-R_2+R_3}\begin{bmatrix}1&1&1\\0&2&-2\\0&0&0\end{bmatrix}$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\to E_1=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}$, $E_2=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_2}\begin{bmatrix}1&1&1\\0&1&-1\\0&0&0\end{bmatrix}\xrightarrow{-R_2+R_1}\begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}=\text{rref}(A)$.

$E_3=\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix}$ $E_4=\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}$

(c) $E_4E_3E_2E_1A=\text{rref}(A)$

$\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&2&-2\\2&4&0\end{bmatrix}$

$=\begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}$.

hwk_w4-2

1) Ch2.2: #19: If A, B, and C are nxn invertible matrices, does the equation $C^{-1}(A+X)B^{-1}=I$ has a solution X? If so, find it.

<u>Solution</u>: $C^{-1}(A+X)B^{-1}=I$

$\Rightarrow C(C^{-1}(A+X)B^{-1})=CI=C$.

$\Rightarrow (A+X)B^{-1}=C$

$\Rightarrow ((A+X)B^{-1})B=CB$

$(A+X)(B^{-1}B)=(A+X)=CB$.

$\Rightarrow \boxed{X=CB-A}$

2) Let $A=\begin{bmatrix}1&0&-2\\-3&1&4\\2&-3&4\end{bmatrix}$.

(a) Use the row operation algorithm from this section to determine if A is invertible. If yes, find the inverse, if not, find rref(A)

(b) Find the elementary matrix for each row operation which are used to obtain rref(A)

(c) If A is invertible, write the inverse A^{-1} as a product of elementary matrices, and write A as a product of elementary matrices as well. If A is not invertible, write A as a product of elementary matrices and rref(A).

<u>Solution</u>. (a) and (b)

$[A\,\vdots\,I]=\left[\begin{array}{ccc|ccc}1&0&-2&1&0&0\\-3&1&4&0&1&0\\2&-3&4&0&0&1\end{array}\right]\xrightarrow{3R_1+R_2}\left[\begin{array}{cccccc}1&0&-2&1&0&0\\0&1&-2&3&1&0\\2&-3&4&0&0&1\end{array}\right]$

$E_1=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}$

$\xrightarrow[E_2]{-2R_1+R_3}\begin{bmatrix}1&0&-2&1&0&0\\0&1&-2&3&1&0\\0&-3&8&-2&0&1\end{bmatrix}\xrightarrow[E_3]{3R_2+R_3}\begin{bmatrix}1&0&-2&1&0&0\\0&1&-2&3&1&0\\0&0&2&7&3&1\end{bmatrix}$

$E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}$ $E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}$

$\xrightarrow[E_4]{R_3+R_2}\begin{bmatrix}1&0&-2&1&0&0\\0&1&0&10&4&1\\0&0&2&7&3&1\end{bmatrix}\xrightarrow[E_5]{R_3+R_1}\begin{bmatrix}1&0&0&8&3&1\\0&1&0&10&4&1\\0&0&2&7&3&1\end{bmatrix}$

$E_4=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$ $E_5=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|ccc}1&0&0&8&3&1\\0&1&0&10&4&1\\0&0&1&7/2&3/2&1/2\end{array}\right]=[I\,\vdots\,A^{-1}]$.

$E_6=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/2\end{bmatrix}$

(a) A is invertible and $A^{-1}=\begin{bmatrix}8&3&1\\10&4&1\\7/2&3/2&1/2\end{bmatrix}$.

(c) $A^{-1}=\begin{bmatrix}8&3&1\\10&4&1\\7/2&3/2&1/2\end{bmatrix}=E_6E_5E_4E_3E_2E_1$

$A=\begin{bmatrix}1&0&-2\\-3&1&4\\2&-3&4\end{bmatrix}=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$

$=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$

hwk_w4-3

Ch2.3: #7, #8. Determine which of the matrices are invertible. Use as few calculations as possible. Justify your answers.

#7 $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$ #8 $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 8 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

Solution, #7, $\text{rref}(\#7) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

#7 matrix is invertible.

#8 is invertible because every column is a pivot column. It is in a row echelon form.

Ch2.3: #11. Mark an implication as True if the truth of "statement 2" always follows whenever "statement 1" happens to be true. An implication is false if there is an instance in which "statement 2" is false but "statement 1" is true. Justify each answer.

a. If the equation $A\vec{x} = 0$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

b. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent.

c. If A is an $n \times n$ matrix, then the equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$

d. If the equation $A\vec{x} = \vec{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

e. If A^T is not invertible, then A is not invertible.

Soly: (a) True, because $\text{rref}(A) = I$, i.e. every column of A is a pivot column.

(b) ~~[illegible]~~ True, because $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^n$ implies rref(A) must be the identity matrix $I_{n \times n}$, and every column of A is a pivot column.

(c) False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\vec{x} = \vec{b}$ has no solution.

(d) True, because A must have a nonpivot column, or a free variable for $A\vec{x} = \vec{0}$.

(e) True, because A is invertible if and only if A^T is invertible. Specifically, $A = E_k \cdots E_1$, a product of elementary matrices. $\Leftrightarrow A^T = E_1^T \cdots E_k^T$, a product of elementary matrices since E_1^T is an elementary matrix.

hwk_w5-1

Ch2.5: #3: Solve $A\vec{x} = \vec{b}$ be LU-factorization, solve $A\vec{x} = \vec{b}$ also by row reduction.

$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$,

$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Soly: $A\vec{x} = \vec{b} \Leftrightarrow \begin{cases} U\vec{x} = \vec{y} \\ L\vec{y} = \vec{b} \end{cases}$

$L\vec{y} = \vec{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \Rightarrow \vec{y} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

$U\vec{x} = \vec{y}$: $\begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

$A\vec{x} = \vec{b}$

$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ -6 & 0 & -2 & 0 \\ 8 & -1 & 5 & 4 \end{array}\right] \rightarrow \text{rref} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$.

Ch2.5: #11: Find an LU factorization of $A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}$

Soly: $A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ -1 & 7 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1+R_3} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 5 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$

$\xrightarrow{-R_2+R_3} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix} = U$.

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

since $(E_3E_2E_1)A = U$, $A = (E_3E_2E_1)^{-1}U$,

$L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}$

So $A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

hwk_w6-1

Ch.3.1: #9: Compute the determinant by cofactor expansion, choosing a row or column at each step that involves the least amount of computation.

Soln:
$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} \overset{R_3}{=} 3\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} \overset{R_1}{=} 3(5)\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= 15(7-6) = 15$$

#13. Same instruction as #9.

Soln:
$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} \overset{R_2}{=} (2)\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$
$$\overset{C_2}{=} (-2)(3)\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} \overset{R_3}{=} (-6)\left(-(-1)\begin{vmatrix} 4 & -5 \\ 5 & -3 \end{vmatrix} + 2\begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}\right) = (-6)[-12+25+2(8-15))$$
$$= 6$$

hwk_w6-2

Ch.3.2. #8: Find the determinant by row reduction to echelon form.

Soln:
$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix} \underset{3R_1+R_4}{\overset{-2R_1+R_3}{=}} \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix}$$
$$\underset{R_2+R_4}{\overset{-R_2+R_3}{=}} \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{vmatrix} \overset{R_3 \leftrightarrow R_4}{=} -\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$
$$= -(1)(1)(1)(10) = -10$$

#27. Mark each statement as True or False. Justify each answer.

a) A row replacement does not affect the determinant of a matrix.

True. By Theorem 3 of Ch.3.2

b) The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during the reduction of A to U.

False. Only true if no row operations of the second type are used.

c). If the columns of A are linearly dependent, then $\det(A) = 0$

True. Because $\det(A) = (-1)^r \det(U)$ where U is a row echelon form of A without using the row operations of the second type, and r is the number of row interchanges used. If the columns of A are linearly dependent, there must be a nonpivot column of U, and there must be a zero row of U. So $\det(U) = 0$ by the zero row expansion and $\det(A) = 0$.

d) $\det(A+B) = \det(A) + \det(B)$

False. Counterexample. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, then $\det(A+B) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$, but $\det(A) = 1$, and $\det(B) = 0$. So $\det(A+B) \neq \det(A) + \det(B)$.

hwk_w6-3

Ch4.1, #20: The set of all continuous real-valued functions defined on a closed interval $[a,b]$ in $\mathbb{R}$ is denoted by $C[a,b]$. This set is a subspace of all real-valued functions defined on $[a,b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a,b]$ is indeed a subspace as claimed?

b. Show that $\{f \text{ in } C[a,b] : f(a)=f(b)\}$ is a subspace of $C[a,b]$.

<u>Solution</u>: a. Let f, g in $C[a,b]$. Need to show (i) $f+g$ is in $C[a,b]$, and (ii) kf is in $C[a,b]$. For (i), because $h(x) := f(x)+g(x) = (f+g)(x)$ is a continuous function in $[a,b]$, (i) holds. Similarly, $h(x) := kf(x) = (kf)(x)$ is also continuous in $[a,b]$, kf is in $C[a,b]$, for any k in $\mathbb{R}$. Obviously, the zero function, $f(x) \equiv 0$ is continuous, thus in $C[a,b]$.

b. Let $H = \{f \text{ in } C[a,b] : f(a)=f(b)\}$. Then for f, g in H, $f(a)=f(b)$ and $g(a)=g(b)$. So ~~[illegible]~~ for $h = f+g$, we have $h(a) = f(a)+g(a) = f(b)+g(b) = h(b)$, and h is in H. Similarly, for any constant k in $\mathbb{R}$, $h = kf$, then $h(a) = kf(a) = kf(b) = h(b)$, so h is in H. This shows H is a subspace of $C[a,b]$.

#24, Mark each statement as True or False and justify each answer.

(a) A vector is any element of a vector space. <u>True</u>. According to definition.

(b) If $\vec{u}$ is a vector in a vector space V ~~then~~ $(-1)\vec{u}$ is the same as the negative of $\vec{u}$. True, (3) on page 193.

(c) A vector space is also a subspace. True, because the addition and scalar multiplication are already closed.

(d) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$. False, because $\begin{bmatrix} a \\ b \end{bmatrix}$ is not element of $\mathbb{R}^3$. But, $\mathbb{R}^2$ is isomorphic to a subspace of $\mathbb{R}^3$.

(e) A subset H of a vector space V is a subspace of V if the following conditions are satisfied: (i) the zero vector of V is in H; (ii) $\vec{u}, \vec{v}$, and $\vec{u}+\vec{v}$ are in H; (iii) c is a scalar and $c\vec{u}$ is in H. True, by definition on p195.

hwk_w7-1

Ch4.2, #24: Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\vec{w}$ is in $Col(A)$. Is $\vec{w}$ in $Nul(A)$?

<u>Sol</u>: $\vec{w}$ in $Col(A)$ iff $\vec{w} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$, or $\vec{w} = [a_1, a_2, a_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\vec{x}$.

Because $[A : \vec{w}] = \begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix}$

$\sim \text{rref}[A : \vec{w}] = \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$A\vec{x} = \vec{w}$ has a solution, and $\vec{w}$ is in $col(A)$. Moreover, $\vec{w} = -\frac{1}{2}\vec{a}_1 + \vec{a}_2$.

To determine if $\vec{w}$ is in $Nul(A)$, just check if $A\vec{w} = \vec{0}$:

$$\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Yes $\vec{w}$ in $Nul(A)$.

#31: Define $T: \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. For instance, if $p(t) = 3+5t+7t^2$, then $T(p) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

(a) Show that T is a linear transformation.

(b) Find a polynomial p in $\mathbb{P}_2$ that spans the ~~[illegible]~~ kernel of T, and describe the range of T.

<u>Sol</u>: (a) Let $p(t) = a_0 + a_1 t + a_2 t^2$ and $q(t) = b_0 + b_1 t + b_2 t^2$, then $(p+q)(t) = (a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2$, and $kp(t) = (ka_0) + (ka_1)t + (ka_2)t^2$. So, $T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix}$

$= \begin{bmatrix} a_0+b_0 \\ (a_0+b_0)+(a_1+b_1)+(a_2+b_2) \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0+a_1+a_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_0+b_1+b_2 \end{bmatrix}$

$= T(p) + T(q)$. Also $T(kp) = \begin{bmatrix} kp(0) \\ kp(1) \end{bmatrix} = \begin{bmatrix} ka_0 \\ k(a_0+a_1+a_2) \end{bmatrix}$

$= \begin{bmatrix} ka_0 \\ ka_0+ka_1+ka_2 \end{bmatrix} = kT(p)$. So $T: \mathbb{P}_2 \to \mathbb{R}^2$ is linear.

(b) For p in kernel(T), $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

$= \begin{bmatrix} a_0 \\ a_0+a_1+a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so, $a_0 = 0$, $a_1 = -a_2$.

and $p(t) = a_0 + a_1 t + a_2 t^2 = a_1(t - t^2)$.

So kernel(T) = Span$\{t - t^2\}$

Last, range$(T) = \mathbb{R}^2$, because for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$, $T(p) = \begin{bmatrix} a_0 \\ a_0+a_1+a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has solution in a_0, a_1, a_2. □

hwk-w7-2

Ch4.3, #17. Find a basis for the space spanned by

$\vec{v}_1=\begin{bmatrix}8\\9\\-3\\-6\\0\end{bmatrix}, \vec{v}_2=\begin{bmatrix}4\\5\\1\\-4\\4\end{bmatrix}, \vec{v}_3=\begin{bmatrix}-1\\-4\\-9\\6\\-7\end{bmatrix}, \vec{v}_4=\begin{bmatrix}6\\8\\4\\-7\\10\end{bmatrix}, \vec{v}_5=\begin{bmatrix}-1\\4\\11\\-8\\-7\end{bmatrix}$

Solu. By row operations,

$[v_1, v_2, v_3, v_4, v_5] \to \text{rref} = \begin{bmatrix}1 & 0 & 0 & -1/2 & 3\\0 & 1 & 0 & 5/2 & -7\\0 & 0 & 1 & 0 & -3\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$

so a basis is $\mathcal{B}=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$V=\text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_5\}$
$=\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}=\text{Span}\left\{\begin{bmatrix}8\\9\\-3\\-6\\0\end{bmatrix}, \begin{bmatrix}4\\5\\1\\-4\\4\end{bmatrix}, \begin{bmatrix}-1\\-4\\-9\\6\\-7\end{bmatrix}\right\}$.

#33. Consider the polynomials $p_1(t)=1+t^2$ and $p_2(t)=1-t^2$. Is $\{p_1, p_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

Solu. Yes. Solve the vector equation

$x_1 p_1(t) + x_2 p_2(t) \equiv \vec{0}$ (*)

to have

$x_1(1+t^2)+x_2(1-t^2) \equiv 0$

$(x_1+x_2)+(x_1-x_2)t^2 \equiv 0$.

Since $\{1, t, t^2\}$ is linearly independent in $\mathbb{P}_3$, we have

$\begin{cases} x_1+x_2=0 \\ x_1-x_2=0 \end{cases}$

for which the trivial solution $x_1=x_2=0$ is the only solution because matrix $A=\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ is invertible. Therefore, (*) has the only trivial solution $x_1=x_2=0$, and by definition, $\{p_1, p_2\}$ is linearly independent in $\mathbb{P}_2$.

hwk-w7-3

Ch4.4, #13: The set $\mathcal{B}=\{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t)=3+t-6t^2$ relative to $\mathcal{B}$.

Solu: Let $[p]_{\mathcal{B}}=\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. Then

$p(t)=x_1 b_1(t)+x_2 b_2(t)+x_3 b_3(t)$.

$[p]_{\mathcal{E}}=x_1[b_1]_{\mathcal{E}}+x_2[b_2]_{\mathcal{E}}+x_3[b_3]_{\mathcal{E}}$
$=[[b_1]_{\mathcal{E}}, [b_2]_{\mathcal{E}}, [b_3]_{\mathcal{E}}][p]_{\mathcal{B}}$
$=P_{\mathcal{B}}\vec{x}$

where $\mathcal{E}=\{1, t, t^2\}$ is the standard basis. So

$[p]_{\mathcal{E}}=\begin{bmatrix}3\\1\\-6\end{bmatrix}=P_{\mathcal{B}}[p]_{\mathcal{B}}=\begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\1 & 1 & 1\end{bmatrix}\vec{x}$

Solve the equation by

$[P_{\mathcal{B}} \vdots [p]_{\mathcal{E}}]=\left[\begin{array}{ccc:c}1 & 0 & 1 & 3\\0 & 1 & 2 & 1\\1 & 1 & 1 & -6\end{array}\right] \xrightarrow{E.R.O} \text{rref}$

$=\begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & -9\\0 & 0 & 1 & 5\end{bmatrix}$.

So $[p]_{\mathcal{B}}=\vec{x}=\begin{bmatrix}-2\\-9\\5\end{bmatrix}$. We can check

$p(t)=3+t-6t^2$
$=-2(1+t^2)-9(t+t^2)+5(1+2t+t^2)$

Ch4.4, #36. Let $H=\text{Span}\{v_1, v_2, v_3\}$ and $\mathcal{B}=\{v_1, v_2, v_3\}$. Show that $\mathcal{B}$ is a basis for H and $\vec{x}$ is in H, and find the $\mathcal{B}$ coordinate of $\vec{x}$, for

$v_1=\begin{bmatrix}-6\\4\\-9\\4\end{bmatrix}, v_2=\begin{bmatrix}8\\-3\\7\\-3\end{bmatrix}, v_3=\begin{bmatrix}-9\\5\\-8\\3\end{bmatrix}, x=\begin{bmatrix}4\\7\\-8\\3\end{bmatrix}$

Solu: By elementary row operations

$[v_1, v_2, v_3, x]=\begin{bmatrix}-6 & 8 & -9 & 4\\4 & -3 & 5 & 7\\-9 & 7 & -8 & -8\\4 & -3 & 3 & 3\end{bmatrix} \to \text{rref} =$

$=\begin{bmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 5\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$. Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are pivot column vectors, they are linearly independent. So $\mathcal{B}=\{v_1, v_2, v_3\}$ is a basis for $H=\text{Span}\{v_1, v_2, v_3\}$. Since x is a nonpivot column, and

$\vec{x}=3\vec{v}_1+5\vec{v}_2+2\vec{v}_3$

$\vec{x}$ is in H, and the combinate weights give the coordinate

$[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}3\\5\\2\end{bmatrix}$.

hwk_w8_1

Ch4.5, #14: Determine the dimensions of Nul(A) and Col(A), where

$$A=\begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Soln</u>: Since A is a ref form, column 1, 3, 4 are pivot columns, and column 2, 5, 6 are nonpivot columns. So

$\dim(\mathrm{Col}(A)) = 3 = \#$ of pivot columns.

$\dim(\mathrm{Nul}(A)) = 3 = \#$ of nonpivot columns

Ch4.5, #21: The first four Hermite polynomials are 1, $2t$, $-2+4t^2$ and $-12t+8t^3$. Show that they form a basis of $\mathbb{P}_3$.

<u>Soln</u>: By the coordinate theorem, Theorem 8 of Ch.4.4, we only need to show that

$[1]_{\mathcal{E}}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $[2t]_{\mathcal{E}}=\begin{bmatrix}0\\2\\0\\0\end{bmatrix}$,

$[-2+4t^2]_{\mathcal{E}}=\begin{bmatrix}-2\\0\\4\\0\end{bmatrix}$, $[-12t+8t^3]_{\mathcal{E}}=\begin{bmatrix}0\\-12\\0\\8\end{bmatrix}$

are linearly independent, where $\mathcal{E}=\{1, t, t^2, t^3\}$ is the standard basis for $\mathbb{P}_3$. By elementary row operations,

$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ is already a ref matrix with all columns pivot. They are linearly independent, forming a basis for $\mathbb{R}^4$, and by Theorem 8, the 4 Hermite polynomials form a basis for $\mathbb{P}_3$.

hwk_w8_2

Ch.4.6, #4: Assume A is row equivalent to B. Without calculations, list rank(A), dim(Nul(A)). Then find bases for Col(A), Row(A), Nul(A)

$$A=\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix}, B=\begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & 1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Soln</u>: $\mathrm{rank}(A) = 3$, # of nonzero rows of ref(A) = B.

$\dim(\mathrm{Nul}(A)) = n-\mathrm{rank}(A) = 6-3 = 3$

Basis of Col(A) $=\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}=\left\{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\-3\\-2\end{bmatrix}, \begin{bmatrix}7\\10\\1\\-5\\0\end{bmatrix}\right\}$

Basis of Row(A) $=\{B_1, B_2, B_3\}$
$=\{[1\ 1\ -3\ 7\ 9\ -9], [0\ 1\ 1\ 3\ 4\ -3], [0\ 0\ 0\ 1\ -1\ -2]\}$.

By row operations

$$B \to \mathrm{rref}(B)=\begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \begin{aligned} x_1 - 2x_3 + 9x_5 + 2x_6 &= 0 \\ x_2 - x_3 + 7x_5 + 3x_6 &= 0 \\ x_4 - x_5 - 2x_6 &= 0 \end{aligned} \Rightarrow \vec{x} = \underbrace{\begin{bmatrix}2\\1\\1\\0\\0\\0\end{bmatrix}}_{=v_1} x_3 + \underbrace{\begin{bmatrix}-9\\-7\\0\\1\\1\\0\end{bmatrix}}_{=v_2} x_5 + \underbrace{\begin{bmatrix}-2\\-3\\0\\2\\0\\1\end{bmatrix}}_{=v_3} x_6$

Basis of Nul(A) $=\{v_1, v_2, v_3\}$
$=\left\{\begin{bmatrix}2\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-9\\-7\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-3\\0\\2\\0\\1\end{bmatrix}\right\}$.

Ch.4.6, #35 Let $A=\begin{bmatrix} 7 & -9 & -4 & 5 & 3 & -3 & -7 \\ -4 & 6 & 7 & -2 & -6 & -5 & 5 \\ 5 & -7 & -6 & 5 & -6 & 2 & 8 \\ -3 & 5 & 8 & -1 & -7 & -4 & 8 \\ 6 & -8 & 5 & 4 & 4 & 9 & 3 \end{bmatrix}$

a. Construct matrices C and N whose columns are bases for Col(A) and Nul(A). Construct a matrix R whose rows form a basis for Row(A).

<u>Soln</u>: $\mathrm{rref}(A)=\begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So. $C=[\vec{a}_1, \vec{a}_2, \vec{a}_4, \vec{a}_6]=\begin{bmatrix} 7 & -9 & 5 & -3 \\ -4 & 6 & -2 & -5 \\ 5 & -7 & 5 & 2 \\ -3 & 5 & -1 & -4 \\ 6 & -8 & 4 & 9 \end{bmatrix}$

Solution for $A\vec{x}=\vec{0}$ is

$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix}=\begin{bmatrix}-13/2\\-11/2\\1\\0\\0\\0\\0\end{bmatrix}x_3+\begin{bmatrix}-5\\-1/2\\0\\11/2\\1\\0\\0\end{bmatrix}x_5+\begin{bmatrix}3\\-2\\0\\-7\\0\\-1\\1\end{bmatrix}x_7 \Rightarrow N=\begin{bmatrix} 13 & -10 & 3 \\ 11 & -1 & -2 \\ -2 & 0 & 0 \\ 0 & 11 & -7 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$R=\begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

b. Construct a matrix M whose columns form a basis for Nul(A^T), form the matrices $S=[R^T\ N]$ and $T=[C\ M]$, and explain why S and T should be square. Verify that both S and T are invertible.

Solution to #35 continues.

$$A^T \xrightarrow{E.R.O's} \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2/11 \\ 0 & 1 & 0 & 0 & -41/11 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 28/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 41 \\ 0 \\ -28 \\ 11 \end{bmatrix} t, \quad M = \begin{bmatrix} 2 \\ 41 \\ 0 \\ -28 \\ 11 \end{bmatrix}.$$

$$S = [R^T N] = \begin{bmatrix} 1 & 0 & 0 & 0 & 13 & -11 & 3 \\ 0 & 1 & 0 & 0 & 11 & -1 & -2 \\ 13/2 & 1/2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 11 & -7 \\ 5 & 1/2 & -1/2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -3 & 2 & 7 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$T = [C\ M] = \begin{bmatrix} 7 & -9 & 5 & -3 & 2 \\ -4 & 6 & -2 & -5 & 41 \\ 5 & -7 & 9 & 2 & 0 \\ -3 & 5 & -1 & -4 & -28 \\ 6 & -8 & 4 & 9 & 11 \end{bmatrix}.$$

Since $\text{rank}(A) + \dim(\text{Nul}(A)) = n$, $n = 7$, $\text{rank}(A) = 4$, $\dim(\text{Nul}(A)) = 3$, $R = R_{4\times 7}$, $R^T = (R^T)_{7\times 4}$, $N = N_{7\times 3}$, so $S = [R^T, N] = S_{7\times 7}$. Also, $C = C_{5\times 4}$
$\text{rank}(A^T) + \dim(\text{Nul}(A^T)) = m = 5$,
$\text{rank}(A^T) = \text{rank}(A) = 4$, $\dim(\text{Nul}(A^T)) = 1$
$\Rightarrow M = M_{5\times 1}$. And $T = [C\ M] = T_{5\times 5}$.
S and T are invertible because $\det(S) = 41241/2 \neq 0$, $\det(T) = 10360 \neq 0$.

hwk. wk 3

Ch 4.7, #7: let $\mathcal{B} = \{b_1, b_2\}$, $C = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$:

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Soln: $P_{\mathcal{B}} = [[b_1]_\mathcal{E}, [b_2]_\mathcal{E}] = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}$

$P_{\mathcal{C}} = [[c_1]_\mathcal{E}, [c_2]_\mathcal{E}] = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}.$

We know
$\text{rref}[P_\mathcal{C} \vdots P_\mathcal{B}] = [I \vdots P_{\mathcal{C}\leftarrow\mathcal{B}}]$
$\text{rref}[P_\mathcal{B} \vdots P_\mathcal{C}] = [I \vdots P_{\mathcal{B}\leftarrow\mathcal{C}}]$.

By elementary row operations
$$[P_\mathcal{C}, P_\mathcal{B}] \to \text{rref} = \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$
$$[P_\mathcal{B}, P_\mathcal{C}] \to \text{rref} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{bmatrix}.$$

we have
$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

Ch 4.7, #13: In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $p = p(t) = -1 + 2t$

Soln: $P_{\mathcal{E}\leftarrow\mathcal{B}} = P_\mathcal{B} = [[b_1]_\mathcal{E}\ [b_2]_\mathcal{E}\ [b_3]_\mathcal{E}]$
$$= \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}.$$

Since $[p]_\mathcal{E} = P_{\mathcal{E}\leftarrow\mathcal{B}} [p]_\mathcal{B}$, we have
$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad [p]_\mathcal{B} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Solve by augmented matrix:
$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \to \text{rref} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$\Rightarrow [p]_\mathcal{B} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$. Check the answer

$$p = -1 + 2t \overset{\checkmark}{=} 5(1-2t+t^2) - 2(3-5t+4t^2) + (2t+3t^2)$$

hwk_w9_1

Ch.5.1 #9: Find a basis for the eigenspace of $\lambda=1,5$ for $A=\begin{bmatrix}5&0\\2&1\end{bmatrix}$.

Solu. For $\lambda=1$, solve $(A-\lambda I)\vec{x}=\vec{0}$, by

$[A-\lambda I]=\begin{bmatrix}4&0\\2&0\end{bmatrix}\to\begin{bmatrix}1&0\\0&0\end{bmatrix}\Rightarrow x_1=0,\ \vec{x}=\begin{bmatrix}0\\1\end{bmatrix}x_2$

$E_{\lambda=1}=\text{Span}\{\begin{bmatrix}0\\1\end{bmatrix}\}$.

· For $\lambda=5$, solve $(A-\lambda I)\vec{x}=\vec{0}$, by

$[A-\lambda I]=\begin{bmatrix}0&0\\2&-4\end{bmatrix}\to\begin{bmatrix}1&-2\\0&0\end{bmatrix},\ x_1=2x_2$

$\Rightarrow\vec{x}=\begin{bmatrix}2\\1\end{bmatrix}x_2,\ E_{\lambda=5}=\text{Span}\{\begin{bmatrix}2\\1\end{bmatrix}\}$.

#15: Find a basis for the eigenspace of eigenvalue $\lambda=3$ for $A=\begin{bmatrix}4&2&3\\-1&1&-3\\2&4&9\end{bmatrix}$

Solu: Solve $(A-\lambda I)\vec{x}=\vec{0}$ by row operations

$A-\lambda I=\begin{bmatrix}1&2&3\\-1&-2&-3\\2&4&6\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow x_1+2x_2+3x_3=0$

$\Rightarrow\vec{x}=\begin{bmatrix}-2\\1\\0\end{bmatrix}x_2+\begin{bmatrix}-3\\0\\1\end{bmatrix}x_3,\ E_{\lambda=3}=\text{Span}\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\}$

#25: Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}

Proof: Let $\vec{v}$ be an eigenvector of λ for A. Then $A\vec{v}=\lambda\vec{v}$. Since $\vec{v}\neq 0$, λ must not be zero. Otherwise, if $\lambda=0$, then $A\vec{v}=\lambda\vec{v}=\vec{0}$, implying A is not invertible for having a nontrivial solution $\vec{v}\neq\vec{0}$ to the homogeneous equation $A\vec{v}=\vec{0}$. Now, $\lambda\neq 0$, $\frac{1}{\lambda}$ is defined, and A is invertible, A^{-1} is defined. Therefore

$$A\vec{v}=\lambda\vec{v}$$
$$\Leftrightarrow\ \vec{v}=A^{-1}(\lambda\vec{v})=\lambda A^{-1}\vec{v}$$
$$\Leftrightarrow\ \tfrac{1}{\lambda}\vec{v}=A^{-1}\vec{v}\ \text{ or }\ A^{-1}\vec{v}=\tfrac{1}{\lambda}\vec{v}$$

So by definition, $\vec{v}\overset{\neq 0}{}$ is an eigenvector of A^{-1} and $\frac{1}{\lambda}$ is the corresponding eigenvalue.

hwk_w9-2

Ch 5.2 #9: Find the characteristic equation of $A=\begin{bmatrix}1&0&-1\\2&3&-1\\0&6&0\end{bmatrix}$.

Solu: $|A-\lambda I|=\begin{vmatrix}1-\lambda&0&-1\\2&3-\lambda&-1\\0&6&-\lambda\end{vmatrix}\ \overset{-R_1+R_2}{\underset{-\lambda R_1+R_3}{=}}\ \begin{vmatrix}1-\lambda&0&-1\\1+\lambda&3-\lambda&0\\\lambda^2-\lambda&6&0\end{vmatrix}$

$=(-1)\begin{vmatrix}1+\lambda&3-\lambda\\\lambda^2-\lambda&6\end{vmatrix}=-[6(1+\lambda)-(\lambda^2-\lambda)(3-\lambda)]$

$=-(\lambda^3-4\lambda^2+9\lambda+6)=0$

#13: Find the characteristic equation of

$A=\begin{bmatrix}6&-2&0\\-2&9&0\\5&8&3\end{bmatrix}$.

Solu: $|A-\lambda I|=\begin{vmatrix}6-\lambda&-2&0\\-2&9-\lambda&0\\5&8&3-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}6-\lambda&-2\\-2&9-\lambda\end{vmatrix}$

$=(3-\lambda)((6-\lambda)(9-\lambda)-4)=(3-\lambda)(50-15\lambda+\lambda^2)$

$=(3-\lambda)(10-\lambda)(5-\lambda)=0\ \Rightarrow\lambda=3,5,10.$

#24. Show that if A and B are similar then $\det A=\det B$.

Proof: A and B are similar if there is an invertible matrix P so that

$$A=PBP^{-1}$$

Because $\det(P^{-1})=\frac{1}{\det(P)}$, we have

$$\det(A)=\det(PBP^{-1})=\det(P)\det(B)\det(P^{-1})$$
$$=\det(P)\cdot\frac{1}{\det(P)}\det(B)=\det(B).$$

hwk_w9-3

ch5.3 #10: Diagonalize $A=\begin{bmatrix}2&3\\4&1\end{bmatrix}$ if possible.

Solution: · Eigenvalues: $\det(A-\lambda I)=0$

$\begin{vmatrix}2-\lambda&3\\4&1-\lambda\end{vmatrix}=(2-\lambda)(1-\lambda)-12=\lambda^2-3\lambda-10=(\lambda+2)(\lambda-5)$
$=0 \Rightarrow \lambda_1=-2, \lambda_2=5 \Rightarrow$ diagonalizable.

· Eigenvectors: Solve for $\lambda=\lambda_1$, $(A-\lambda I)\vec{x}=\vec{0}$ by $[A-\lambda_1 I]=\begin{bmatrix}4&3\\4&3\end{bmatrix}\to\begin{bmatrix}4&3\\0&0\end{bmatrix}$, $4x_1+3x_2=0$.

Pick one nontrivial solution for eigenvector $\vec{v}_1=\begin{bmatrix}3\\-4\end{bmatrix}$.

Solve for $\lambda=\lambda_2$, $(A-\lambda I)\vec{x}=\vec{0}$, by
$[A-\lambda_2 I]=\begin{bmatrix}-3&3\\4&-4\end{bmatrix}\to\begin{bmatrix}1&-1\\0&0\end{bmatrix}$, $x_1-x_2=0$

Pick one eigenvector solution: $\vec{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$.

So $P=[\vec{v}_1\ \vec{v}_2]=\begin{bmatrix}3&1\\-4&1\end{bmatrix}$, $D=\begin{bmatrix}-2&0\\0&5\end{bmatrix}$,

$A=\begin{bmatrix}2&3\\4&1\end{bmatrix}=PDP^{-1}=\begin{bmatrix}3&1\\-4&1\end{bmatrix}\begin{bmatrix}-2&0\\0&5\end{bmatrix}\begin{bmatrix}3&1\\-4&1\end{bmatrix}^{-1}$

#16: Diagonalize $A=\begin{bmatrix}0&-4&-6\\-1&0&-3\\1&2&5\end{bmatrix}$, with given eigenvalues $\lambda=2,1$.

Soln: · For $\lambda=2$, solve $(A-\lambda I)\vec{x}=\vec{0}$ by
$[A-\lambda I]=\begin{bmatrix}-2&-4&-6\\-1&-2&-3\\1&2&3\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow x_1+2x_2+3x_3=0$

$\vec{x}=\begin{bmatrix}-2\\1\\0\end{bmatrix}x_2+\begin{bmatrix}-3\\0\\1\end{bmatrix}x_3$. Eigenvectors: $\vec{v}_1=\begin{bmatrix}-2\\1\\0\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-3\\0\\1\end{bmatrix}$

($\Rightarrow$ diagonalizable since $\dim(E_{\lambda=2})+\dim(E_{\lambda=1})=3$)

· For $\lambda=1$, solve $(A-\lambda I)\vec{x}=\vec{0}$ by
$[A-\lambda I]=\begin{bmatrix}-1&-4&-6\\-1&-1&-3\\1&2&4\end{bmatrix}\to\begin{bmatrix}1&2&4\\0&1&1\\0&-2&-2\end{bmatrix}\to\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\end{bmatrix}$

~~$\lambda=1 \Rightarrow$ not an eigenvalue.~~

~~Solve $\det(A-\lambda I)=\ldots=(2-\lambda)^2(1-\lambda)=0 \Rightarrow \lambda=2, 1$~~

~~For $\lambda=-1$, solve $(A-\lambda I)\vec{x}=\vec{0}$ by~~

$\Rightarrow \begin{matrix}x_1+2x_3=0\\x_2+x_3=0\end{matrix} \Rightarrow \vec{x}=\begin{bmatrix}-2\\-1\\1\end{bmatrix}x_3$, $\vec{v}_3=\begin{bmatrix}-2\\-1\\1\end{bmatrix}$.

$P=[\vec{v}_1,\vec{v}_2,\vec{v}_3]=\begin{bmatrix}-2&-3&-2\\1&0&-1\\0&1&1\end{bmatrix}$, $D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}$.

for $A=PDP^{-1}$ or $AP=PD$.

#29: A factorization $A=PDP^{-1}$ is not unique. Demonstrate this for A in Example 2. With $D_1=\begin{bmatrix}3&0\\0&5\end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A=P_1D_1P_1^{-1}$.

Soln. $A=\begin{bmatrix}7&2\\-4&1\end{bmatrix}$. It is given $P=\begin{bmatrix}1&1\\-1&-2\end{bmatrix}$ and $D=\begin{bmatrix}5&0\\0&3\end{bmatrix}$ with $A=PDP^{-1}$. Then for $D_1=\begin{bmatrix}3&0\\0&5\end{bmatrix}$, $P_1=\begin{bmatrix}1&1\\-2&-1\end{bmatrix}$ to keep the correspondence between the eigenvalues of A from D_1 to column vectors of P_1 for eigenvectors. □.

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hwk_w10-1

Ch.5.4, #5: Let $T:\mathbb{P}_2\to\mathbb{P}_3$ be the transformation that maps a polynomial $p(t)$ into the polynomial $(t+5)p(t)$.

a. Find the image of $p(t)=2-t+t^2$

b. Show that T is a linear transformation.

c. Find the matrix for T relative to the bases $\{1,t,t^2\}$ and $\{1,t,t^2,t^3\}$

Soln: a. $T(2-t+t^2)=(t+5)(2-t+t^2)$
$=10-3t+4t^2+t^3$

b. Because $T(p+q)=(t+5)(p(t)+q(t))$
$=(t+5)p(t)+(t+5)q(t)=T(p)+T(q)$,
and $T(kp)=(t+5)(kp(t))=k(t+5)p(t)$
$=kT(p)$.

c. Let $p(t)=a_0+a_1t+a_2t^2$, $[p]_{\mathcal{E}}=\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$.

$T(p)=(t+5)(a_0+a_1t+a_2t^2)=5a_0+(a_0+5a_1)t+(a_1+5a_2)t^2+a_2t^3$. So

$[T(p)]_{\mathcal{E}}=\begin{bmatrix}5a_0\\a_0+5a_1\\a_1+5a_2\\a_2\end{bmatrix}=\begin{bmatrix}5&0&0\\1&5&0\\0&1&5\\0&0&1\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$
$=[T][p]_{\mathcal{E}}$. So the matrix of T relative to the standard bases is

$[T]=\begin{bmatrix}5&0&0\\1&5&0\\0&1&5\\0&0&1\end{bmatrix}:\mathbb{R}^3\to\mathbb{R}^4$.

#13: Define $T:\mathbb{R}^2\to\mathbb{R}^2$ by $T(x)=Ax$ with $A=\begin{bmatrix}0&1\\-3&4\end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

Soln: · Eigenvalues of A: $\det(A-\lambda I)$
$=\begin{vmatrix}-\lambda&1\\-3&4-\lambda\end{vmatrix}=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0$
$\lambda_{1,2}=1,3$. · Eigenvectors for $\lambda=\lambda_1=1$,
$A-\lambda_1I=\begin{bmatrix}-1&1\\-3&3\end{bmatrix}\to\begin{bmatrix}1&-1\\0&0\end{bmatrix}$, $x_1=x_2$, $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}x_2$
$\Rightarrow \vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$. For $\lambda=\lambda_2=3$, $A-\lambda_2I=\begin{bmatrix}-3&1\\-3&1\end{bmatrix}$
$\to\begin{bmatrix}3&-1\\0&0\end{bmatrix}\Rightarrow 3x_1-x_2=0$, $\vec{x}=\begin{bmatrix}1\\3\end{bmatrix}x_1$, $\vec{v}_2=\begin{bmatrix}1\\3\end{bmatrix}$.

Let $\mathcal{B}=\{\vec{v}_1,\vec{v}_2\}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$. Then

$[T(\vec{v}_1)]_{\mathcal{B}}=[\lambda_1\vec{v}_1]_{\mathcal{B}}=\begin{bmatrix}\lambda_1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$,

$[T(\vec{v}_2)]_{\mathcal{B}}=[\lambda_2\vec{v}_2]_{\mathcal{B}}=\begin{bmatrix}0\\\lambda_2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$,

and $[T]_{\mathcal{B}}=[\,[T(\vec{v}_1)]_{\mathcal{B}},[T(\vec{v}_2)]_{\mathcal{B}}\,]=\begin{bmatrix}1&0\\0&3\end{bmatrix}=D$.

hwk_w10_2

Ch5.5 #9: Let $A=\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$. The transformation $x \mapsto Ax$ is the composition of a rotation and a scaling. Give the angle φ ~~and~~ of the rotation, where $-\pi<\varphi<\pi$ and give the scale factor r

Soln. Let $a=-\sqrt{3}/2$, $b=-1/2$, and $r=\sqrt{a^2+b^2}=1$, $\tan\varphi=\frac{b}{a}=\frac{1}{\sqrt{3}}$.
φ is in the 3rd quadrant.
Since $\sin(-\pi+\frac{\pi}{6})=-1/2$, $\cos(-\pi+\frac{\pi}{6})=-\sqrt{3}/2$,
$\varphi=-\pi+\frac{\pi}{6}=-\frac{5\pi}{6}$. So

$$A=r\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}=1\cdot\begin{bmatrix} \cos(-\frac{5\pi}{6}), & -\sin(-\frac{5\pi}{6}) \\ \sin(-\frac{5\pi}{6}), & \cos(-\frac{5\pi}{6}) \end{bmatrix}$$

a rotation by $-\frac{5\pi}{6}$ only, without scaling.

#16: Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A=PCP^{-1}$ where $A=\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

Soln. $\cdot\ |A-\lambda I|=\begin{vmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}=\lambda^2-8\lambda+17=0$

$\lambda_{1,2}=\frac{-(-8)\pm\sqrt{8^2-4(1)(17)}}{2(1)}=\frac{8\pm\sqrt{-4}}{2}=4\pm i$

$\cdot$ For $\lambda=\lambda_1=4-i$, solve $(A-\lambda_1 I)\vec{x}=\vec{0}$ by

$[A-\lambda_1 I]=\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\xrightarrow[-(1+i)R_2+R_1]{}\begin{bmatrix} 0 & 0 \\ 1 & -1+i \end{bmatrix}$

$=\begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}$, $x_1+(-1+i)x_2=0$, $\vec{x}=\begin{bmatrix} 1-i \\ 1 \end{bmatrix}x_2$.

$\vec{v}_1=\begin{bmatrix} 1-i \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}+\begin{bmatrix} -1 \\ 0 \end{bmatrix}i$. $\vec{\alpha}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{\beta}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$P=[\vec{\alpha},\vec{\beta}]=\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $P^{-1}=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$,

$C=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}=\sqrt{17}\begin{bmatrix} 4/\sqrt{17} & -1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix}$

$=\sqrt{17}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, $\varphi=\tan^{-1}\frac{1}{4}$

$$A=\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}=PCP^{-1}=\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Rmk. Let $T(\vec{x})=A\vec{x}$. Let $\mathcal{B}=\{\vec{\alpha},\vec{\beta}\}$ be a basis of $\mathbb{R}^2$. Then with respect to $\mathcal{B}$, the matrix of transformation of T is
$[T]_{\mathcal{B}}=C=\sqrt{17}R_{\varphi}$, a composition of scaling $\sqrt{17}$ and a rotation by $\varphi=\tan^{-1}\frac{1}{4}$.

hwk_w11

Ch6.1 #19: Mark each statement True or False. Justify each answer

(a). $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2$. True, because $\|\vec{v}\|=\sqrt{\vec{v}\cdot\vec{v}}$.

(b) For any scalar c, $\vec{u}\cdot(c\vec{v})=c(\vec{u}\cdot\vec{v})$
True, because Theorem 1(c).

(c) If the distance from $\vec{u}$ to $\vec{v}$ equals the distance from $\vec{u}$ to $-\vec{v}$, then $\vec{u}$ and $\vec{v}$ are orthogonal. True, by Figure 5, p335 and its explanation.

(d) For a square matrix A, vectors in $Col(A)$ are orthogonal to vectors in $Nul(A)$.
False, $(Col(A))^{\perp}=Nul(A^T)$ by Thm3 and $A^T\neq A$ in general.

(e) If vectors $\vec{v}_1,\cdots,\vec{v}_p$ span a subspace W and if $\vec{x}$ is orthogonal to each $\vec{v}_j$, $j=1,\cdots,p$, then $\vec{x}$ is in $W^{\perp}$.
True, by 1. in the box of p336.

#29: Let $W=\text{Span}\{\vec{v}_1,\cdots,\vec{v}_p\}$. Show that if $\vec{x}$ is orthogonal to each $\vec{v}_j$, $j=1,2,\cdots,p$, then $\vec{x}$ is orthogonal to every vector in W.

Solution. By assumption, $\vec{x}\cdot\vec{v}_j=0$ for $j=1,2,\cdots,p$. For each $\vec{v}$ in W, there are $c_1,c_2,\cdots,c_p$ so that

$$\vec{v}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_p\vec{v}_p.$$

then

$$\begin{aligned}\vec{x}\cdot\vec{v}&=\vec{x}\cdot(c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_p\vec{v}_p)\\&=\vec{x}\cdot(c_1\vec{v}_1)+\vec{x}\cdot(c_2\vec{v}_2)+\cdots+\vec{x}\cdot(c_p\vec{v}_p)\\&=c_1(\vec{x}\cdot\vec{v}_1)+c_2(\vec{x}\cdot\vec{v}_2)+\cdots+c_p(\vec{x}\cdot\vec{v}_p)\\&=c_1(0)+c_2(0)+\cdots+c_p(0)\\&=0\end{aligned}$$

that is, $\vec{x}$ is orthogonal to $\vec{v}$. $\square$

hwk_w11-2

Ch 6.2: #5: Determine if the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \right\}$ is orthogonal.

Solu: $\vec{v}_1 \cdot \vec{v}_2 = 3(-1) + (-2)(3) + 1(-3) + 3(4) = 0$

$\vec{v}_1 \cdot \vec{v}_3 = 3(3) + (-2)8 + 1(7) + 3(0) = 0$

$\vec{v}_2 \cdot \vec{v}_3 = (-1)(3) + 3(8) + (-3)7 + 4(0) = 0.$

Yes, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal.

#13: Let $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\vec{y}$ as the sum of two orthogonal vectors, one in Span$\{\vec{u}\}$ and one orthogonal to $\vec{u}$.

Solu: $\vec{w} = \text{proj}_{\vec{u}} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{-13}{65} \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$

$\vec{w}^{\perp} = \text{perp}_{\vec{u}} \vec{y} = \vec{y} - \text{proj}_{\vec{u}} \vec{y}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{-13}{65} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 182 \\ 104 \end{bmatrix} / 65$

So $\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{w} + \vec{w}^{\perp} = \begin{bmatrix} -52 \\ 91 \end{bmatrix} / 65 + \begin{bmatrix} 182 \\ 104 \end{bmatrix} / 65$

#29: Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix.

Solution: U is orthogonal if and only if $U^T U = U U^T = I$ for $n \times n$ matrices. Let $W = UV$. Then $W^T W = (UV)^T (UV)$ $= (V^T U^T)(UV) = V^T (U^T U) V = V^T I V$ $= V^T V = I$. Similarly, $W W^T = (UV)(UV)^T$ $= (UV)(V^T U^T) = U (V V^T) U^T = U I U^T$ $= U U^T = I$. So W is orthogonal.

hwk_w12-1

Ch 6.3 #7: Let $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$. Write $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W.

Solu: $\{\vec{u}_1, \vec{u}_2\}$ is orthogonal because $\vec{u}_1 \cdot \vec{u}_2 = 1(5) + 3(1) + (-2)4 = 0$. So

$\vec{w} = \text{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$

$= \frac{0}{14} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{28}{42} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 70 \\ 14 \\ 56 \end{bmatrix} / 21$

$\vec{w}^{\perp} = \text{perp}_W \vec{y} = \vec{y} - \text{proj}_W \vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 70 \\ 14 \\ 56 \end{bmatrix} / 21 = \begin{bmatrix} -49 \\ 49 \\ 49 \end{bmatrix} / 21$

So $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \vec{w} + \vec{w}^{\perp} = \begin{bmatrix} 70 \\ 14 \\ 56 \end{bmatrix} / 21 + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \frac{49}{21}$

#17. Let $\vec{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$.

(a) Let $U = [\vec{u}_1, \vec{u}_2]$. Compute $U^T U$ and $U U^T$

(b) Compute $\text{proj}_W \vec{y}$ and $(U U^T) \vec{y}$

Solu. $\{\vec{u}_1, \vec{u}_2\}$ is orthonormal because $\vec{u}_1 \cdot \vec{u}_2 = 0$, $\vec{u}_1 \cdot \vec{u}_1 = 1$, $\vec{u}_2 \cdot \vec{u}_2 = 1$. So

(a) $U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$, $U^T U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}^T \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$U U^T = \frac{1}{9} \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$

(b) $\text{proj}_W \vec{y} = (\vec{y} \cdot \vec{u}_1) \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \vec{u}_2$

$= \frac{18}{3} \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + \frac{9}{3} \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 18 \\ 36 \\ 45 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$,

$(U U^T) \vec{y} = \frac{1}{9} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 18 \\ 36 \\ 45 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$

So $\text{proj}_W \vec{y} = (U U^T) \vec{y}$ if $\{\vec{u}_1, \cdots, \vec{u}_p\}$ is orthonormal and $W = \text{Span}\{\vec{u}_1, \cdots, \vec{u}_k\}$

hwk-W12-2

Ch6.4 #9: Find an orthogonal basis for the column space of $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix} = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$

Soln: $\cdot\ \vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$, $W_1 = \text{Span}\{\vec{v}_1\}$.

$\cdot\ \vec{v}_2 = \text{perp}_{W_1} \vec{a}_2 = \vec{a}_2 - \dfrac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \dfrac{-40}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $W_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$\cdot\ \vec{v}_3 = \text{perp}_{W_2} \vec{a}_3 = \vec{a}_3 - \dfrac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$

$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \dfrac{30}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \dfrac{-10}{20} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

is an orthogonal basis for Col(A).

#12. Find an orthogonal basis for Col(A) where $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix} = [\vec{a}_1, \vec{a}_2, \vec{a}_3]$

Soln: $\cdot\ \vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $W_1 = \text{Span}\{\vec{a}_1\}$

$\cdot\ \vec{v}_2 = \text{perp}_{W_1} \vec{a}_2 = \vec{a}_1 - \dfrac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \dfrac{16}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$W_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{a}_1, \vec{a}_2\}$.

$\vec{v}_3 = \text{perp}_{W_2} \vec{a}_3 = \vec{a}_3 - \dfrac{a_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \dfrac{a_3 \cdot v_2}{v_2 \cdot v_2} \vec{v}_2$

$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \dfrac{14}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \dfrac{12}{8} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \sim \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

is an orthogonal basis for Col(A).

hwk-W12-3

Ch6.5 #5: Find all least-square solutions of the equation $A\vec{x} = \vec{b}$, where

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

Soln: let $\hat{A} = A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

and $\hat{b} = A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$. Solve

$\hat{A}\hat{x} = \hat{b}$ by

$[\hat{A}\ \hat{b}] = \left[\begin{array}{ccc:c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \to \begin{bmatrix} 2 & 1 & 1 & 7 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 5 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref} \Rightarrow \begin{matrix} x_1 + x_3 = 5 \\ x_2 - x_3 = -3 \end{matrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$

All solutions are on the line through point $\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$ in the direction of $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

#11: let $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Find the orthogonal projection of $\vec{b}$ to Col(A), (b) a least-squares solution of $A\vec{x} = \vec{b}$.

Soln. Since the columns of A, $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are orthogonal, $\vec{a}_1 \cdot \vec{a}_2 = \vec{a}_1 \cdot \vec{a}_3 = \vec{a}_2 \cdot \vec{a}_3 = 0$,

$\text{proj}_{\text{Col}(A)} \vec{b} = \dfrac{\vec{b} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \dfrac{\vec{b} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \dfrac{\vec{b} \cdot \vec{a}_3}{\vec{a}_3 \cdot \vec{a}_3} \vec{a}_3$

$= \dfrac{36}{54} \begin{bmatrix} 4 \\ 1 \\ 6 \\ 1 \end{bmatrix} + \dfrac{0}{27} \begin{bmatrix} 0 \\ -5 \\ 1 \\ -1 \end{bmatrix} + \dfrac{9}{27} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$

(b) let $\hat{A} = A^T A = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$

$\hat{b} = A^T \vec{b} = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$. Solve

$\hat{A}\hat{x} = \hat{b}$ by

$[\hat{A}, \hat{b}] = \begin{bmatrix} 54 & 0 & 0 & 36 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 27 & 9 \end{bmatrix} \to \hat{x} = \begin{bmatrix} 36/54 \\ 0 \\ 9/27 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$

for which

$A\hat{x} = \frac{2}{3}\vec{a}_1 + 0\,\vec{a}_2 + \frac{1}{3}\vec{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix} = \text{proj}_{\text{Col}(A)} \vec{b}$