Name:______________________________ Score:__________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(15pts) A force of 1 newton stretches a spring 0.5 meters. A mass of 2kg is attached to the spring horizontally on a flat surface. Assume the force due to friction is proportional to the velocity of the mass with the proportionality equal to 4N-sec/m.

(a) Write an initial value problem for the mass-spring problem if the mass is released at a position that the spring is stretched 10 cm.

(b) Determine the motion of the mass by solving the IVP.

2(15pts) Find the solution to the initial value problem

$$\begin{cases} x'' + 3x' + 3x = 0 \\ x(0) = 1, \ x'(0) = -1 \end{cases}$$

3(10pts) The roots for the characteristic (auxiliary) equation of a 4th order linear, homogeneous equation with constant coefficients $a_4x^{(4)} + \ldots + a_1x' + a_0x = 0$ are

$$-2, \ 2 \pm 2i.$$

Find a general solution to the equation.

4(20pts) Determine the form of a particular solution for each equation

(a) $x''' + 2x'' + 2x' = 0$

(b) $x''' + 2x'' + 2x' = t + 5e^{-t}\sin t$

(c) $x''' + 2x'' + 2x' = (t^2 + 1)e^{-t}$

5(20pts) Find a particular solution to the nonhomogeneous equation

$$x'' + 2x' - 8x = te^t + e^{-t}.$$

6(20pts) Assume $y_1(x) = e^x, \ y_2(x) = x + 1$ are two linearly independent solutions to the homogeneous part of the nonhomogeneous equation

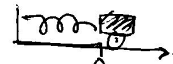
$$xy'' - (x+1)y' + y = x^2.$$

(a) Find a particular solution to the nonhomogeneous equation.

(b) Find a solution to the initial value problem with $xy'' - (x+1)y' + y = x^2, \ y(1) = -1, y'(1) = -2.$

END

Soln Key Math221 Test 3 Summer(2), '02

1 (15pts): $1N = k(0.5) \Rightarrow k = 2N/m$  , $b = 4N\,sec/m$, $k = 2kg$

(a) $mx'' + bx' + kx = 0 \Rightarrow 2x'' + 4x' + 2x = 0$ with $x(0) = 0.1m$, $x'(0) = 0$

(c) $\begin{cases} x'' + 2x' + x = 0 \\ x(0) = 0.1,\ x'(0) = 0 \end{cases}$ $r^2 + 2r + 1 = (r+1)^2 = 0$, $x(t) = c_1 e^{-t} + c_2 t e^{-t}$, $0.1 = c_1 + c_2 0$

$0 = x'(0) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}|_{t=0} = -c_1 + c_2$

$\Rightarrow c_1 = 0.1,\ c_2 = c_1 = 0.1 \Rightarrow x(t) = 0.1e^{-t} + 0.1te^{-t} = 0.1(t+1)e^{-t}$

2 (15pts) $\begin{cases} x'' + 3x' + 3x = 0 \\ x(0) = 1,\ x'(0) = -1, \end{cases}$ $r^2 + 3r + 3 = 0$, $r_{1,2} = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm i\sqrt{3}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$

$x(t) = c_1 e^{-3/2 t}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-3/2 t}\sin\frac{\sqrt{3}}{2}t$. $1 = x(0) = c_1$, $-1 = x'(0) = c_1\left(-\frac{3}{2}e^{-\frac{3}{2}t}\cos\frac{\sqrt{3}}{2}t + e^{-3/2 t}\left(-\frac{\sqrt{3}}{2}\right)\sin\frac{\sqrt{3}}{2}t\right) + c_2\left(-\frac{3}{2}e^{-\frac{3}{2}t}\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}e^{-\frac{3}{2}t}\cos\frac{\sqrt{3}}{2}t\right)\Big|_{t=0} = -\frac{3}{2}c_1 + \frac{\sqrt{3}}{2}c_2$

$\Rightarrow c_1 = 1,\ c_2 = \left(-1 + \frac{3}{2}c_1\right)\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow x(t) = e^{-3/2 t}\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{-\frac{3}{2}t}\sin\frac{\sqrt{3}}{2}t$

3 (10pts) -2 must be a double root. Hence $x(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 e^{2t}\cos 2t + c_4 e^{2t}\sin 2t$

4 (20pts) $x''' + 2x'' + 2x' = 0$. $r^3 + 2r^2 + 2r = r(r^2 + 2r + 2) = 0$, $r_1 = 0$, $r_{2,3} = -1 \pm i$

(a) $g(t) = 0 \Rightarrow x_p(t) = 0$ (b) $g(t) = t + 5e^{-t}\sin t$, $x_p(t) = At^2 + Bt + Cte^{-t}\sin t + Dte^{-t}\cos t$

(c) $g(t) = (t^2+1)e^{-t}$, $x_p(t) = (At^2 + Bt + C)e^{-t}$

5 (20pts) $x'' + 2x' - 8x = te^t + e^{-t}$ solve $x'' + 2x' - 8x = 0$, $r^2 + 2r - 8 = (r+4)(r-2) = 0$

$r_1 = -4,\ r_2 = 2,\ x_1(t) = e^{-4t},\ x_2(t) = e^{2t}$. $g(t) = te^t + e^{-t} = g_1(t) + g_2(t)$.

For $g_1(t)$, $x_{p_1}(t) = (At + B)e^t$. $x_{p_1}' = (At + A + B)e^t$, $x_{p_1}'' = (At + 2A + B)e^t$

$x_{p_1}'' + 2x_{p_1}' - 8x_{p_1} = ((1+2-8)At + 2A + B + 2(A+B) - 8B)e^t = te^t$.

$\Rightarrow -5A = 1$, $A = -\frac{1}{5}$, $4A - 5B = 0$. $B = \frac{4}{5}A = -\frac{4}{25}$. $x_{p_1}(t) = -\left(\frac{1}{5}t + \frac{4}{25}\right)e^{t}$

For $g_2(t)$, $x_{p_2}(t) = Ae^{-t}$, $x_{p_2}' = -Ae^{-t}$, $x_{p_2}'' = Ae^{-t}$, $x_{p_2}'' + 2x_{p_2}' - 8x_{p_2} = (A - 2A - 8A)e^{-t} = e^{-t}$

$\Rightarrow A = -\frac{1}{9}$. $\Rightarrow x_{p_2}(t) = -\frac{1}{9}e^{-t}$. $\Rightarrow x_p(t) = -\left(\frac{1}{5}t + \frac{4}{25}\right)e^t - \frac{1}{9}e^{-t}$

6 (20pts) (a) $W[y_1, y_2] = \begin{vmatrix} e^x & x+1 \\ e^x & 1 \end{vmatrix} = e^x - e^x(x+1) = -xe^x$ $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$u_1(x) = -\int \frac{g y_2}{aW}dx = -\int \frac{x^2(x+1)}{x\cdot(-xe^x)}dx = \int (xe^{-x} + e^{-x})dx = -xe^{-x} - 2e^{-x}$

$u_2(x) = \int \frac{g y_1}{aW}dx = \int \frac{x^2 e^x}{x(-xe^x)}dx = -\int dx = -x \Rightarrow y_p(x) = (-x-2) + (-x)(x+1)$

$= -2(x+1) - x^2$

(b) general soln. $y(x) = c_1 e^x + c_2(x+1) - 2(x+1) - x^2 = c_1 e^x + c_2(x+1) - x^2$.

$-1 = y(1) = c_1 e + 2c_2 - 1 \Rightarrow c_1 e + 2c_2 = 0$. $-2 = y'(1) = c_1 e^x + c_2 - 2x|_{x=1} = c_1 e + c_2 - 2$

$\Rightarrow c_1 e + c_2 = 0 \Rightarrow c_1 = c_2 = 0. \Rightarrow y(x) = -x^2$

End