July 12, 2002 **Math 221 Test 5** **Summer 2002**

Name:______________________________ Score:________

Instructions: You must show supporting work to receive full and partial credits. No text book, personal notes, formula sheets allowed. One formula sheet will be provided.

1(**10pts**) Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

(a) Find a general solution of the system.

(b) Sketch a phase portrait of the system.

2(**10pts**) Sketch a phase portrait of this system of two competing species

$$\begin{cases} x' = x(2 - x - 2y) \\ y' = y(2 - y - 2x) \end{cases}$$

by including its nullclines, typical vector fields on and off the nullclines, separatrix if any, and a few typical solution curves. Is there a coexisting equilibrium state? Explain why or why not.

3(**10pts**) Use definition *only* to find the Laplace transform of the unit step function

$$u_a(t) = \begin{cases} 0, \ t < a \\ 1, \ t \geq a \end{cases}$$

Using any other method receives no credit.

4(**20pts**) Find the Laplace transforms of these functions

(a) $(e^t + t)^2$

(b) $f(t) = \begin{cases} 2, & t < 1 \\ 2 - t, & 1 \leq t < 3, \\ 0, & t \geq 3 \end{cases}$ which must be first expressed in terms of some unit step functions.

(c) $t \sin 2t$

5(**25pts**) Find the Laplace inverses of these functions

(a) $\dfrac{2s + 1}{2s^2 + 4s + 4}$

(b) $\left(\dfrac{1}{s} + e^{-2s}\right)^2$

(c) $\dfrac{s^2 + 2s + 2}{s^4 + s^3 + s^2}$

6(**15pts**) Solve the initial value problem: $\begin{cases} x'' + x = \delta_2(t) \\ x(0) = 0, \ x'(0) = 1. \end{cases}$

7(**10pts**) Fill in the following blanks and justify your answers:

(a) If $\mathcal{L}\{f(t)\}(s) = \dfrac{\sqrt{s}}{(s+1)^2}$, then $\mathcal{L}\{e^{-2t} f(t)\} =$ ____________.

(b) If $\mathcal{L}^{-1}\{F(s)\}(t) = \dfrac{1}{t^2 + 1}$, then $\mathcal{L}^{-1}\{e^{-s} F(s)\}(t) =$ ____________.

Bonus 8pts: Use the Laplace transformation method to solve this system of equations: $\begin{cases} x' = y - t \\ y' = -x + 1 \\ x(0) = 0, \ y(0) = 1. \end{cases}$

Warning : No partial credit will be given. Don't try if you don't have time to waste.

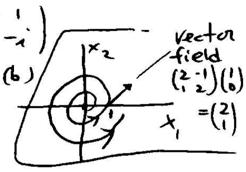
The End

Math 221 Test 5 Sol'n. Key Summer 02 (1st S)

1 (10 pts) (a) $\det|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ +1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2+1=0 \Rightarrow \lambda_{1,2} = 2\pm i$

$(A-\lambda_1 I)\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=0 \Rightarrow -u_1 i - u_2 = 0.\ u_2 = -iu_1 \Rightarrow \vec{u} = \begin{pmatrix}1\\-i\end{pmatrix}$

$\begin{pmatrix}1\\-i\end{pmatrix}e^{(2+i)t} = e^{2t}\begin{pmatrix}\cos t + i\sin t\\ \sin t - i\cos t\end{pmatrix} \Rightarrow \boxed{\vec{x}(t) = c_1e^{2t}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + c_2e^{2t}\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}}$

(b) 

2 (10 pts) $x' = x(2-x-2y)$ x-nullcline: $x=0,\ 2-x-2y=0$

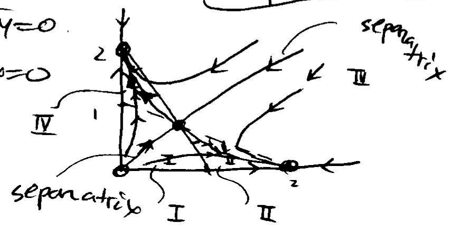
$y' = y(2-y-2x)$ y-nullcline: $y=0,\ 2-y-2x=0$

	I	II	III	IV
$\dot{x}$	+	+	−	−
$\dot{y}$	+	−	−	+

~~nontrivial~~ equilibriums

$\begin{cases}2-x-2y=0\\ 2-y-2x=0\end{cases} \Rightarrow \begin{matrix}x = 2/3\\ y = 2/3\end{matrix}$

and $(2,0)$, $(0,2)$, $(0,0)$.



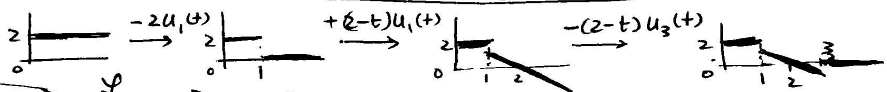
No. For $(x(0),y(0))$ below the separatrix, y-species dies out. For $(x(0),y(0))$ above the separatrix, x-species dies out.

3 (10 pts) $\mathcal{L}\{u_a(t)\}(s) = \int_0^\infty e^{-st}u_a(t)\,dt = \int_a^\infty e^{-st}dt = \lim_{A\to\infty} -\frac{1}{s}e^{-st}\Big|_a^A$

$= \frac{e^{-as}}{s} - \lim_{A\to\infty}\frac{1}{s}e^{-sA} = \boxed{\frac{e^{-as}}{s}}$ provided $s>0$.

4 (20 pts) (a) $(e^t+t)^2 = e^{2t}+2te^t+t^2 \xrightarrow{\mathcal{L}} \boxed{\frac{1}{s-2}+2\frac{1}{(s-1)^2}+\frac{2}{s^3}}$

(b) $f(t) = \begin{cases}2, & t<1\\ 2-t, & 1\le t<3\\ 0, & t\ge 3\end{cases}$



$= 2 - tu_1(t) - (2-t)u_3(t) \xrightarrow{\mathcal{L}} \frac{2}{s} - e^{-s}\mathcal{L}\{t+1\} - e^{-3s}\mathcal{L}\{2-(t+3)\}$

$= \boxed{\frac{2}{s} - e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) + e^{-3s}\left(\frac{1}{s^2}+\frac{1}{s}\right)}$

(c) $\mathcal{L}\{t\sin 2t\} = -\frac{d}{ds}\mathcal{L}\{\sin 2t\}$

$= -\frac{d}{ds}\frac{2}{s^2+4} = \frac{4s}{(s^2+4)^2}$

5 (25 pts) (a) $\frac{2s+1}{2s^2+4s+4} = \frac{1}{2}\left(\frac{2s+1}{s^2+2s+2}\right) = \frac{1}{2}\left(\frac{2(s+1-1)+1}{(s+1)^2+1}\right) = \frac{1}{2}\left[\frac{2(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$ $\xrightarrow{\mathcal{L}^{-1}} \boxed{e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t}$

(b) $\left(\frac{1}{s}+e^{-2s}\right)^2 = \frac{1}{s^2}+2\frac{1}{s}e^{-2s}+e^{-4s} \xrightarrow{\mathcal{L}^{-1}} \boxed{t+2u_2(t)+\delta_4(t)}$

(c) $\frac{s^2+2s+2}{s^4+s^3+s^2} = \frac{s^2+2s+2}{s^2(s^2+s+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+s+1} = \frac{A(s^3+s^2+s)+B(s^2+s+1)+s^2(Cs+D)}{s^2(s^2+s+1)}$

$\Rightarrow A+C=0,\ A+B+D=1,\ A+B=2,\ B=2 \Rightarrow B=2, A=0, C=0, D=-1$

$\Rightarrow \frac{s^2+2s+2}{s^4+s^3+s^2} = \frac{2}{s^2} - \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} \xrightarrow{\mathcal{L}^{-1}} \boxed{2t - \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}$

6 (15 pts) $\mathcal{L}\{x''\}+\mathcal{L}\{x\} = \mathcal{L}\{\delta_2(t)\} = e^{-2s} \to s^2\mathcal{L}\{x\}-sx(0)-x'(0)+\mathcal{L}\{x\} = (s^2+1)\mathcal{L}\{x\}-1 = e^{-2s}$

$\Rightarrow \mathcal{L}\{x\} = \frac{1}{s^2+1}+e^{-2s}\frac{1}{s^2+1}$, $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\} = \boxed{\sin t + u_2(t)\sin(t-2)}$

7 (10 pts). (a) $\mathcal{L}\{e^{2t}f(t)\}(s) = \mathcal{L}\{f(t)\}(s)\big|_{s\to s-2} = \frac{\sqrt{s-2}}{(s-1)^2}$ (b) $\mathcal{L}^{-1}\{e^{-s}F(s)\}(t) = u_1(t)\frac{1}{(t-1)^2+1}$

Bonus: $\begin{cases}s\mathcal{L}\{x\}-x(0) = \mathcal{L}\{y\}-\frac{1}{s^2}, & X=\mathcal{L}\{x\}\\ s\mathcal{L}\{y\}-y(0) = -\mathcal{L}\{x\}+\frac{1}{s}, & Y=\mathcal{L}\{y\}\end{cases} \to \begin{cases}sX = Y-\frac{1}{s^2}\\ sY-1 = -X+\frac{1}{s}\end{cases} \to \begin{cases}Y = sX+\frac{1}{s^2}\\ s(sX+\frac{1}{s^2})-1 = -X+\frac{1}{s}\end{cases} \to s^2X+X-1=0.\ X = \frac{1}{s^2+1}$

$\to Y = \frac{s}{s^2+1}+\frac{1}{s^2} \Rightarrow x(t) = \mathcal{L}^{-1}\{X\} = \sin t,\ \ y(t) = \mathcal{L}^{-1}\{Y\} = \cos t + t$

(End)