Name:______________________________ Score:________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(**20pts**) Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} -5 & -6 \\ 3 & 4 \end{bmatrix}$

(a) Find a general solution of the system.

(b) Sketch a phase portrait of the system, including all straight line solutions, and a few typical solutions.

2(**15pts**) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

and eigenvectors corresponding to the *complex* eigenvalues only.

3(**15pts**) It is given that $\lambda = -2 + i$ is an eigenvalue of a real valued 2×2 matrix A and $\mathbf{u} = \begin{pmatrix} i \\ 1+i \end{pmatrix}$ is a corresponding eigenvector. Find the solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4(**20pts**) Sketch a phase portrait of the system of equations

$$\begin{cases} x' = y \\ y' = x - y \end{cases}$$

by including its nullclines, typical vector fields on and off the nullclines, separatrix if any, and a few typical solution curves. Must the separatrix solutions be straight lines? why or why not?

5(**15pts**) Consider a cooperative system of two species

$$\begin{cases} x' = x(1 - x + y) \\ y' = y(1 - y + 0.5x) \end{cases}$$

For $x(0) > 0, y(0) > 0$, will the solution converge to a co-existence state? Use a phase plane analysis to answer this question.

6(**15pts**) Consider a competitive system of two species

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(1 - y - 0.5x) \end{cases}$$

Is co-existence possible? Which specie must die out and under what condition? Base your conclusion on a phase plane analysis of the system.

Bonus 5pts: The state flower of Nebraska is ____________________

The End

Math 221 Exam 4 Soln Key Summer 02

1 (20pts) (a) $A=\begin{bmatrix}-5 & -6\\ 3 & 4\end{bmatrix}$. $\det|A-\lambda I| = \begin{vmatrix}-5-\lambda & -6\\ 3 & 4-\lambda\end{vmatrix} = \lambda^2+\lambda-2 = (\lambda-1)(\lambda+2)=0$,
$\lambda_{1,2}=-2, 1$. (b) $\lambda_1=-2$. Solve $(A-\lambda_1 I)\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}-3 & -6\\ 3 & 6\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow u_1+2u_2=0$
$\Rightarrow u_1=-2u_2$. Pick $u_2=1 \Rightarrow \vec{u}_1=\begin{pmatrix}-2\\1\end{pmatrix}$. For $\lambda_2=1$, $(A-\lambda_2 I)\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow$
$\begin{pmatrix}-6 & -6\\ 3 & 3\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow u_1+u_2=0,\ u_1=-u_2 \Rightarrow \vec{u}_2=\begin{pmatrix}-1\\1\end{pmatrix}$.
General solution: $\vec{x}(t)=\boxed{c_1\begin{pmatrix}-2\\1\end{pmatrix}e^{-2t}+c_2\begin{pmatrix}-1\\1\end{pmatrix}e^{t}}$ (b)

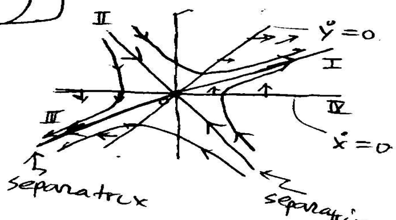
2 (15pts) $|A-\lambda I| = \begin{vmatrix}2-\lambda & 0 & 0\\ 0 & 1-\lambda & 3\\ 2 & -3 & 1-\lambda\end{vmatrix} = (2-\lambda)((1-\lambda)^2+9)=0 \Rightarrow \boxed{\lambda_1=2,\ \lambda_{2,3}=1\pm 3i}$
$(A-\lambda_2 I)\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}-3i & 0 & 0\\ 0 & 3i & 3\\ 2 & -3 & 3i\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}-3iu_1\\ 3iu_2+3u_3\\ 2u_1-3u_2+3iu_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow u_1=0$ and $3iu_2+3u_3=0$ $\Rightarrow u_3=-iu_2$.
$\Rightarrow \vec{u}_1=\boxed{\begin{pmatrix}0\\1\\-i\end{pmatrix}}$. For $\lambda_3=1-3i$, $\vec{u}_2=\boxed{\begin{pmatrix}0\\1\\i\end{pmatrix}}$.

3 (15pts) $Z(t)=\vec{u}e^{(-2+i)t}=\begin{pmatrix}i\\1+i\end{pmatrix}e^{-2t}(\cos t+i\sin t)=\begin{pmatrix}-e^{-2t}\sin t+ie^{-2t}\cos t\\ e^{-2t}(\cos t-\sin t)+ie^{-2t}(\sin t+\cos t)\end{pmatrix}$
General soln: $\vec{x}(t)=c_1e^{-2t}\begin{pmatrix}\sin t\\ \cos t-\sin t\end{pmatrix}+c_2e^{-2t}\begin{pmatrix}\cos t\\ \sin t+\cos t\end{pmatrix}$. $\vec{x}(0)=c_1\begin{pmatrix}0\\1\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}$
$=\begin{pmatrix}c_2\\c_1+c_2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$. $\Rightarrow c_2=1, c_1=-1$. $\Rightarrow \vec{x}(t)=\boxed{e^{-2t}\begin{pmatrix}\cos t-\sin t\\ 2\sin t\end{pmatrix}}$

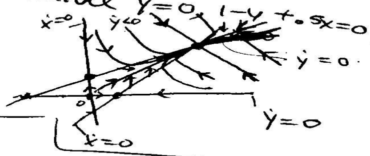
4 (20pts) $\begin{cases}x'=y\\ y'=x-y\end{cases}$ x-nullcline: $y=0$
y-nullcline: $x-y=0$, $y=x$

	I	II	III	IV
$\dot{x}$	+	+	−	−
$\dot{y}$	+	−	−	+
	↗	↘	↙	↖

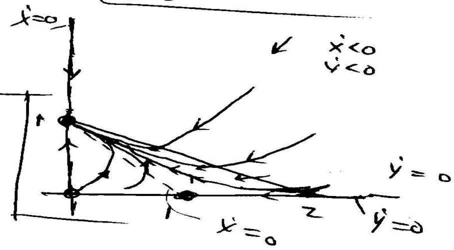
Yes. The separatrix solns are straight line solutions. They are eigensolutions or the eigenvectors are on the straight lines



5 (15pts) $\begin{cases}x'=x(1-x+y)\\ y'=y(1-y+.5x)\end{cases}$, x-nullcline $x=0$, $1-x+y=0$. y-nullcline $y=0$, $1-y+.5x=0$
For $x(0)>0$, $y(0)>0$, the solution converges to a co-existing equilibrium point which is the soln. to $\begin{cases}1-x+y=0\\ 1-y+.5x=0\end{cases} \Rightarrow x=4, y=3$



6 (15pts) $\begin{cases}x'=x(1-x-y)\\ y'=y(1-y-0.5x)\end{cases}$ x-nullcline $x=0$, $1-x-y=0$
y-nullcline $y=0$, $1-y-.5x=0$
(x-species must die out so long as $y(0)>0$)



Bonus: Goddard. but any answer will receive the 5pts.