Name:______________________________ Score:__________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(15pts) Find a general solution to this 1st order linear equation

$$\frac{dx}{dt} = -\frac{t}{t^2+1}x + \frac{t}{t^2+1}.$$

2(15pts) Find the solution to the initial value problem

$$\begin{cases} x'' + 3x' + 2x = 4 \\ x(0) = 1,\ x'(0) = -1 \end{cases}$$

3(10pts) The roots for the characteristic (auxiliary) equation of a 5th order linear, homogeneous equation with constant coefficients $a_5x^{(5)} + \ldots + a_1x' + a_0x = 0$ are

$$-2,\ -2,\ -2,\ 2 \pm 2i.$$

Find a general solution to the equation.

4(20pts) Determine the form of a particular solution for the differential equation

(a) $x'''' + 2x''' + 2x'' = t$

(b) $x'''' + 2x''' + 2x'' = 5e^{-t}\cos t$

(c) $x'''' + 2x''' + 2x'' = 5 + te^{4t}$

5(20pts) Find a particular solution to the nonhomogeneous equation

$$x'' + 5x' + 6x = te^{2t}.$$

6(20pts) $x_1(t) = e^t, x_2(t) = e^t \ln t$ are two linearly independent solutions to the homogeneous part of the nonhomogeneous equation

$$tx'' + (1-2t)x' + (t-1)x = e^t.$$

Find a particular solution to the nonhomogeneous equation.

END

Test 3 Math221 Soln Key Summer 02

1 (15 pts) $\frac{dx}{dt} = -\frac{t}{t^2+1}x + \frac{t}{t^2+1}$. Homogeneous part $\frac{dx}{dt} = -\frac{t}{t^2+1}x$, soln:

$x_h(t) = C\exp\left(\int -\frac{t}{t^2+1}\,dt\right) = C\exp\left(-\frac{1}{2}\ln(t^2+1)\right) = Ce^{\ln\frac{1}{\sqrt{t^2+1}}} = \frac{c}{\sqrt{t^2+1}}$. Particular solution:

$x_p(t) = \frac{C(t)}{\sqrt{t^2+1}}$, $x_p'(t) = \frac{C'(t)}{\sqrt{t^2+1}} - \frac{C(t)\cdot t}{(t^2+1)^{3/2}} = \frac{C'}{\sqrt{t^2+1}} - \frac{t}{t^2+1}\cdot\frac{C(t)}{\sqrt{t^2+1}} \overset{\text{set}}{=} -\frac{t}{t^2+1}x_p + \frac{t}{t^2+1}$,

$\Rightarrow \frac{C'}{\sqrt{t^2+1}} = \frac{t}{t^2+1} \Rightarrow C' = \frac{t}{\sqrt{t^2+1}} \Rightarrow C(t) = \int\frac{t}{\sqrt{t^2+1}}\,dt = \sqrt{t^2+1}$. $\Rightarrow x_p = \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}} = 1$.

$\Rightarrow$ General soln. $x(t) = \boxed{\frac{C}{\sqrt{t^2+1}} + 1}$ (Also by integration factor method as an alternative or formulas for $C(t)$)

2 (15 pts) Solve $x''+3x'+2x=0$. $r^2+3r+2=(r+2)(r+1)=0 \Rightarrow x_1(t)=e^{-2t}$, $x_2(t)=e^{-t}$

Particular solution to $x''+3x'+2x=4$ has form $x_p(t)=A$ so

$x_p''+3x_p'+2x_p = 2A = 4 \Rightarrow A=2$. General soln. $x(t)=c_1e^{-2t}+c_2e^{-t}+2$

Initial conditions: $1=x(0)=c_1+c_2+2$, $-1=x'(0)=-2c_1-c_2$

$\Rightarrow \begin{cases} c_1+c_2=-1 \\ -2c_1-c_2=-1\end{cases} \Rightarrow -c_1=-2,\ c_1=2,\ c_2=-3 \Rightarrow x(t) = \boxed{2e^{-2t}-3e^{-t}+2}$

3 (10 pts) $x(t) = \boxed{c_1e^{-2t}+c_2te^{-2t}+c_3t^2e^{-2t}+c_4e^{2t}\cos 2t+c_5e^{2t}\sin 2t}$

4 (20 pts) · Solve $x''''+2x'''+2x''=0 \Rightarrow r^4+2r^3+2r^2 = r^2(r^2+2r+2)=0$

$\Rightarrow r=0$, double, $-1\pm i \left(=\frac{-2\pm\sqrt{2^2-4(1\times 2)}}{2(1)}\right)$

(a) $g(t)=t$, $x_p(t) = t^s(At+B)$, $s=2$ since $r=0$ is a double root.
$= \boxed{At^3+Bt^2}$

(b) $g(t) = 5e^{-t}\cos t$, $x_p(t) = t^s(Ae^{-t}\cos t + Be^{-t}\sin t)$, $s=1$ since $r=-1+i$ is a simple root
$= \boxed{Ate^{-t}\cos t + Bte^{-t}\sin t}$

(c) $g(t) = 5+te^{4t}$, $x_p(t) = x_{p_1}(t)+x_{p_2}(t) = \boxed{At^2+(Bt+C)e^{4t}}$

5 (20 pts) Solve $x''+5x'+6x=0$, $r^2+5r+6=(r+2)(r+3)=0$, $r_{1,2}=-2,-3$, $x_1(t)=e^{-2t}$, $x_2(t)=e^{-3t}$

Particular solution $x_p(t) = t^s(At+B)e^{2t}$, $s=0$ since $r=\alpha=2$ is not a root.
$= (At+B)e^{2t}$

$x_p'(t) = Ae^{2t}+2(At+B)e^{2t} = (2At+A+2B)e^{2t}$, $x_p''(t) = 2Ae^{2t}+2(2At+A+2B)e^{2t}$
$= (4At+4A+4B)e^{2t}$

$\Rightarrow x_p''+5x_p'+6x_p = [(4A+10A+6A)t + (4A+4B+5A+10B+6B)]e^{2t}$

$= (20At+(9A+20B))e^{2t} = te^{t}$. $\Rightarrow 20A=1$, $A=\frac{1}{20}$, $9A+20B=0$, $B=-\frac{9}{400}$

$\Rightarrow x_p(t) = \boxed{\left(\frac{1}{20}t - \frac{9}{400}\right)e^{2t}}$

6 (20 pts). $x_1(t)=e^t$, $x_2(t)=e^t\ln t$, $W[x_1,x_2] = \begin{vmatrix} e^t & e^t\ln t \\ e^t & e^t\ln t + e^t/t\end{vmatrix} = \frac{e^{2t}}{t}$.

$x_p(t) = v_1(t)x_1(t)+v_2(t)x_2(t)$, $v_1(t) = -\int\frac{gx_2}{aW}\,dt = -\int\frac{e^te^t\ln t}{t\,e^{2t}/t}\,dt = -\int\ln t\,dt$

$v_1(t) = -\int\ln t\,dt = -t\ln t + \int t\cdot\frac{1}{t}\,dt = \boxed{-t\ln t+t}$

$v_2(t) = \int\frac{gx_1}{aW}\,dt = \int\frac{e^te^t}{t\,e^{2t}/t}\,dt = \int dt = t \Rightarrow x_p(t) = (-t\ln t+t)e^t + te^t\ln t$
$= \boxed{te^t}$