**June 14, 2002** **Math 221 Test 1** **Summer 2002**

Name:______________________ PIN(in any 4 digits):__________ Score:__________

**Instructions:** You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(15pts) (a) Verify that $y(x) = \sin \pi x - \ln x$ is the solution to this IVP: $y'' + \pi^2 y = 1/x^2 - \pi^2 \ln x,\ y(1) = 0$.

(b) Does $x^3 + xy + y^3 = 3$ define an implicity solution to the equation $y' + x^2y^2 = x$?

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2(15pts) Find the solution to the initial value problem $\dfrac{dy}{dx} = \dfrac{1+y^2}{yx^2},\ y(1) = 1$.

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3(10pts) At what initial points $(t_0, y_0)$ does the Theorem of Existence and Uniqueness for the initial value problem: $\dfrac{dy}{dt} = \dfrac{2}{t-y},\ y(t_0) = y_0$ apply? Determine the region and verify all conditions of the theorem.

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4(15pts) Consider the IVP: $y' = \dfrac{1}{x}(y^2 + x),\ y(1) = -1$.

(a) Use the Euler method to find approximate values of the solution at these points $x = 1, 1.2, 1.4, 1.6$.

(b) Sketch your approximating solution in the $xy$ plane.

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5(15pts) Consider the linear equation $ty' = -2y + t^2$.

(a) Write it in the standard form $y' = p(t)y + q(t)$ (or $y' + p(t)y = q(t)$).

(b) Find a general solution to the equation.

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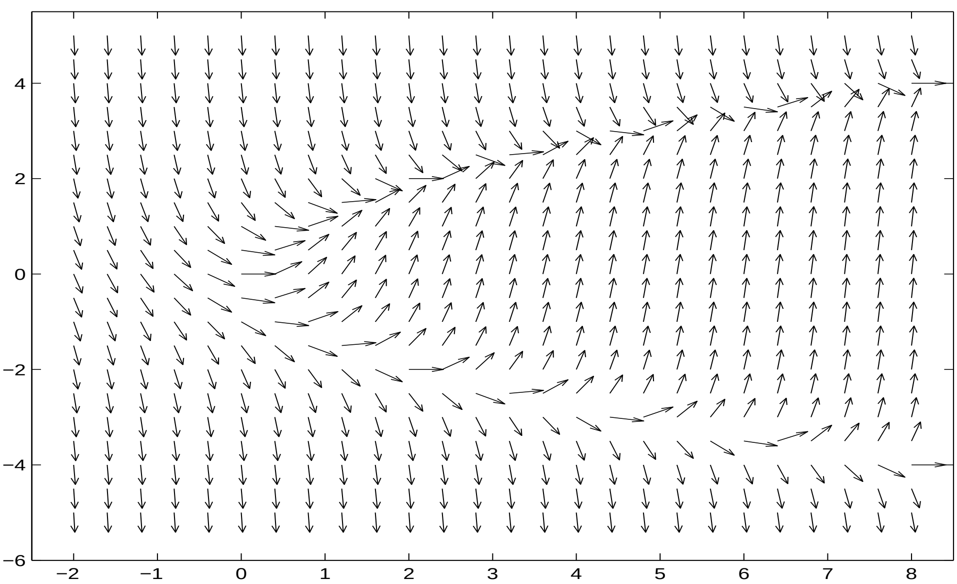
6(15pts) (a) Consider the equation $x' = -\sqrt{x} + t$. Sketch these isoclines: $f(t, x) = -2, -1, 0, 1, 2$ and the vector field on the isoclines in the upper half plane $x > 0$.

(b) The vector field of an equation is given below. Sketch solutions that go through these points: (i) $(-2, 0)$, (ii) $(-1, 1)$.

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7(15pts) Consider the autonomous equation $\dfrac{dy}{dt} = f(y) = y^3 - y$.

(a) Sketch the phase line and classify each equilibrium point as sink, source, or node,

(b) Sketch a qualitative portrait for the solutions, including the ones through $y(0) = -1.5, -0.5, 0.5, 1.5$.

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**END**

# Math 221 Test 1 Soln. Key Summer 2002

**1 (15pts)** (a) $y = \sin \pi x - \ln x$, $y' = \pi \cos \pi x - \frac{1}{x}$, $y'' = -\pi^2 \sin \pi x + \frac{1}{x^2}$

$y'' + \pi^2 y = -\pi^2 \sin \pi x + \frac{1}{x^2} + \pi^2 \sin \pi x - \pi^2 \ln x = \frac{1}{x^2} - \pi^2 \ln x$. $y(1) = \sin \pi - \ln 1 = 0$

(b) $x^3 + xy + y^3 = 3$. $\Rightarrow 3x^2 + y + xy' + 3y^2 y' = 0 \Rightarrow y' = -\frac{3x^2 + y}{x + 3y^2} \neq x - x^2 y^2$

Does not

**2 (15 pts)** $\frac{dy}{dx} = \frac{1+y^2}{yx^2}$, $\frac{y}{1+y^2} dy = \frac{1}{x^2} dx \Rightarrow \frac{1}{2} \ln|1+y^2| = -\frac{1}{x} + c$.

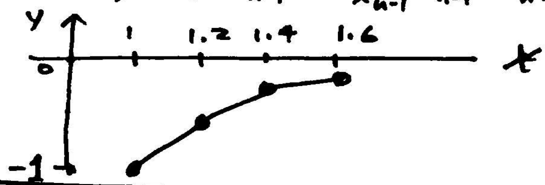
$\Rightarrow 1 + y^2 = e^{-\frac{2}{x} + c} = c' e^{-\frac{2}{x}}$, $y(1) = 1 \Rightarrow c' = 2e^2$

$\Rightarrow y = \sqrt{2e^2 e^{-2/x} - 1}$

**3 (15pts)** $\frac{dy}{dt} = \frac{2}{t-y}$, $f(t,y) = \frac{2}{t-y}$, $\frac{\partial f}{\partial y} = \frac{2}{(t-y)^2}$. Both are continuous in the region $t \neq y$. The theorem applies to any IVP with $t_0 \neq y_0$.

**4 (15pts)** $y' = \frac{1}{x}(y^2 + x)$, $y(1) = -1$. $x_0 = 1$, $y_0 = -1$, $y_n = y_{n-1} + h \frac{1}{x_{n-1}}(y_{n-1}^2 + x_{n-1})$, $h = 0.2$

| $n$ | $x_n$ | $y_n$ | $y_n' = \frac{1}{x_n}(y_n^2 + x_n)$ |
|---|---|---|---|
| 0 | 1 | $-1$ | 2 |
| 1 | 1.2 | $-0.6$ | 1.3 |
| 2 | 1.4 | $-0.34$ | 1.0825714 |
| 3 | 1.6 | $-0.12$ | |




**5 (15pts).** (a) $y' = -\frac{2}{t} y + t$. (b) $y' + \frac{2}{t} y = t$, $\mu(t) = e^{\int \frac{2}{t} dt} = e^{\ln t^2} = t^2$

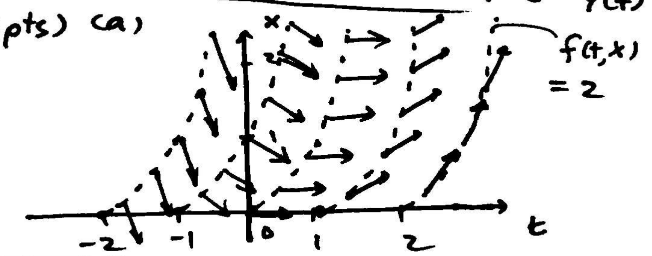
$\rightarrow t^2 y' + \frac{2}{t} y \cdot t^2 = t^3 \Rightarrow (t^2 y)' = t^3 \Rightarrow t^2 y = \int t^3 dt = \frac{1}{4} t^4 + c \Rightarrow y(t) = \frac{c}{t^2} + \frac{1}{4} t^2$

(alternatively: $y' = -\frac{2}{t} y$ has general solution: $y_h(t) = c e^{\int -\frac{2}{t} dt} = ct^{-2}$.

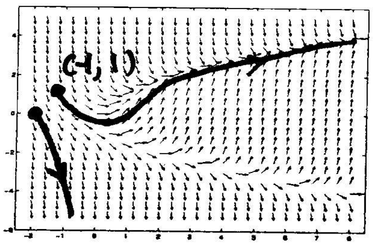
$y' = -\frac{2}{t} y + t$ has a particular solution $y_p(t) = t^{-2} \int t^2 \cdot t \, dt = t^{-2} \frac{1}{4} t^4 = \frac{1}{4} t^2$

$\Rightarrow$ therefore the general soln. is $y(t) = y_h(t) + y_p(t) = ct^{-2} + \frac{1}{4} t^2$ )

**6 (15pts)** (a)




(b)




**7 (15 pts)** (a) $\frac{dy}{dt} = y^3 + y = y(y^2 - 1) = f(y)$.

eq. pt. $y = 0$, sink

$y = \pm 1$, source