EXAMPLE 1 To solve the homogeneous equation with constant coefficients

$$x'' = 3x' - 2x,$$

type at the prompt:

```
> dsolve(diff(x(t),t$2)=3*diff(x(t),t)-2*x(t),x(t));
```

and Maple will display the solution

```
x(t) = _C1 exp(2 t) + _C2 exp(t)
```

i.e., $x(t) = C_1e^{2t} + C_2e^t$.

EXAMPLE 2 Maple also solves initial value problems. If we attach to the equation in Example 1 the initial conditions $x(0) = 2$, $x'(0) = 1$, we can solve the corresponding initial value problem as follows:

```
> dsolve({diff(x(t),t$2)=3*diff(x(t),t)-2*x(t),
  x(0)=2, D(x)(0)=1},x(t));
```

and Maple displays the solution

```
x(t) = - exp(2 t) + 3 exp(t)
```

EXAMPLE 3 Since Maple knows the solution of the above equation, it can solve the nonhomogeneous equation

$$x'' = 3x' - 2x + t.$$

```
> dsolve(diff(x(t),t$2)=3*diff(x(t),t)-2*x(t)+t,x(t));
```

Maple displays the solution

```
x(t) = 1/2 t + 3/4 + _C1 exp(2 t) + _C2 exp(t)
```

EXAMPLE 4 The homogeneous equation with variable coefficients

$$x'' = -(1/t)x' + (4/t^2)x$$

can be solved by typing

```
> dsolve(diff(x(t),t$2)=-(1/t)*diff(x(t),t)+
  (4/t^2)*x(t),x(t));
```

Maple displays the solution

$$x(t) = \frac{\_C1 t^4 + \_C2}{t^2}$$

EXAMPLE 5 The nonhomogeneous equation

$$x'' = -(1/t)x' + (4/t^2)x + t^2$$

can now be solved by typing

```
> dsolve(diff(x(t),t$2)=-(1/t)*diff(x(t),t)+
  (4/t^2)*x(t)+t^2,x(t));
```

after which Maple displays the solution

$$x(t) = 1/12 \frac{t^6 + 12\_C1 t^4 + 12\_C2}{t^2}$$

Numerical solutions

With Maple's help we will first apply Euler's method to solve the initial value problem used in the examples of Section 3.6,

$$x'' = -(2/t)x' + (2/t^2)x, \qquad x(1) = 1, x'(1) = 2, \tag{1}$$

in the interval $[1, 2.5]$, with step size $h = 0.1$. Then we will solve the same initial value problem using the second-order Runge-Kutta method. In each case we write a program, explain how it works, and outline the logic behind it.

Euler's method

```
> f := (t, x, y) → -(2/t) * y + (2/t^2) * x;
```

$$f := (t, x, y) \rightarrow -2\ y/t + 2\frac{x}{t^2}$$

```
> t0:=1: x0:=1: y0:=2:
> h:=0.1:
> n:=15:
> t:=t0: x:=x0: y:=y0:
> for i from 1 to n do
  u:=y:  v:=f(t,x,y):
  x:=x+h*u:  y:=y+h*v:
  t:=t+h:
  print(t,x,y);
  od:
```

and Maple displays the numerical result

```
1.1, 1.2, 1.80
1.2, 1.380, 1.671074380
1.3, 1.547107438, 1.584228650
1.4, 1.705530303, 1.523590803
1.5, 1.857889383, 1.479968678
1.6, 2.005886251, 1.447785244
1.7, 2.150664775, 1.423521952
1.8, 2.293016970, 1.404883714
1.9, 2.433505341, 1.390329781
2.0, 2.572538319, 1.378799518
2.1, 2.710418271, 1.369546482
```

```
> f := (t, x, y) → - (2/t) * y + (2/t^2) * x;
  f := (t, x, y) → - 2 y/t + 2 x/t^2
> t0:=1: x0:=1: y0:=2:
> M0:=[t0,x0]:
> h:=0.1:
> n:=15:
> t:=t0: x:=x0: y:=y0: M:=M0:
> for i from 1 to n do
  a:=y:  b:=f(t,x,y):
  u:=x+h*a:  v:=y+h*b:
  x:=x+h*(a+v)/2:
  y:=y+h*(b+f(t+h,u,v))/2:
  M:=(M,[t,x]):
  t:=t+h:
  od:
  plot([M]);
```

and Maple displays the picture in Figure 3.7.1(a). If we want the graph of $y(t)$, then we need to change `M0:=[t0,x0]:` to `M0:=[t0,y0]:` and `M:=(M,[t,x]):` to `M:=(M,[t,y]):`. If we would like to have the phase-

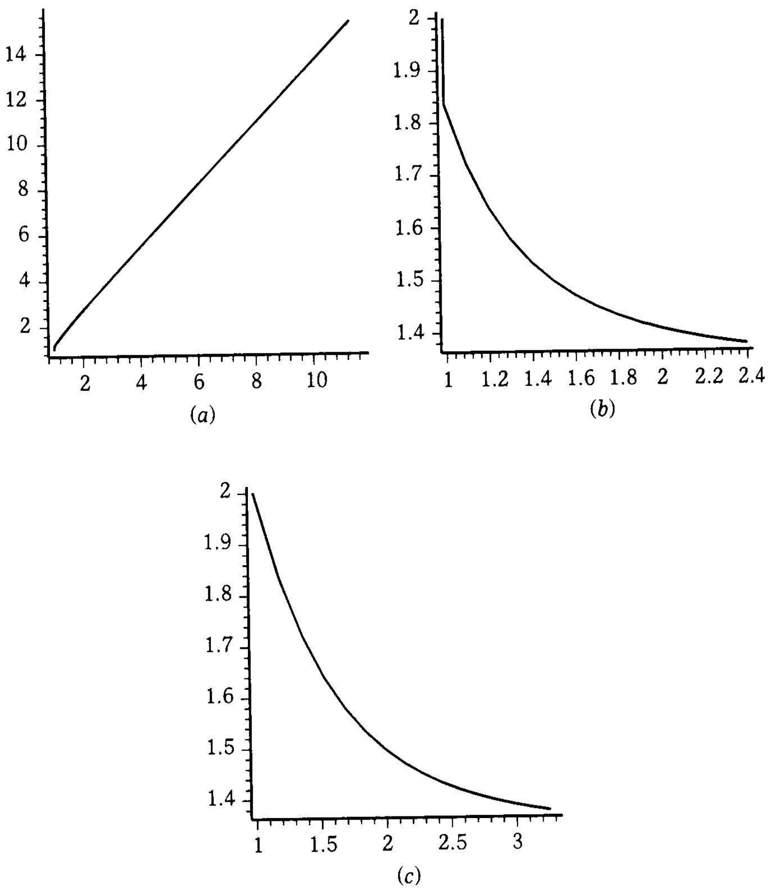
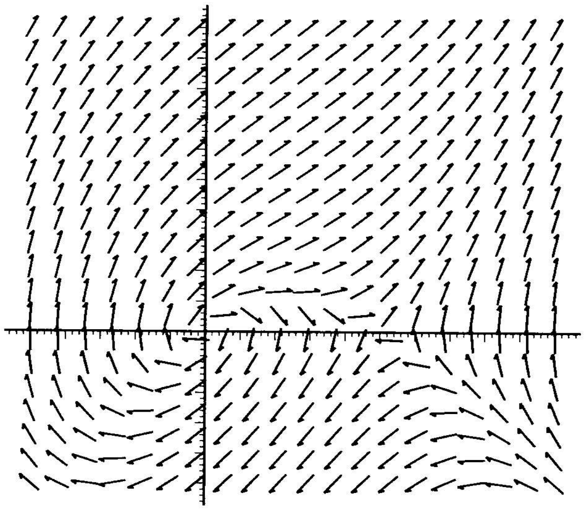


Figure 3.7.1. The graphs (a) of x, (b) of y, and (c) the phase-plane representation of the solution (x, y) for the initial value problem (1).

ᒣ.2. The direction field (3).



plane representation of the solution, then we must replace `M0:=[t0,x0]:` with `M0:=[x0,y0]:` and `M:=(M,[t,x]):` with `M:=(M,[x,y]):`.

Maple also has a built-in program that draws the graph of the solution. It is called `DEplot` and can be used within the package `DEtools`. For more details see Section 4.6 and the on-line help of Maple.

Direction fields

To understand the phase-plane portrait of a second-order equation, it is often useful to draw the direction field. The package `DEtools` has for this purpose a special command called `dfieldplot`. For equation (4) of Section 3.5,

$$x'' = x' + x^2 - x, \tag{2}$$

the direction field can be drawn as follows. We write (2) as a system,

$$\begin{cases} x' = y \\ y' = y + x^2 - x, \end{cases} \tag{3}$$

and proceed as below. Then Maple produces the picture in Figure 3.7.2.

```
> with(DEtools):
> dfieldplot([diff(x(t),t)=y(t), diff(y(t),t)=y(t)+
  (x(t))^2-x(t)],[x(t),y(t)], t=-2..2, x=-1..2, y=-1..2);
```