Math 221 Final Exam **Dec. 17, 2004**

Name:________________________ Score:________

Instructions: You must show supporting work to receive full and partial credits. No text book, personal notes, formula sheets allowed.

1(**18 pts**) Find the general solution to the linear equation $t\dfrac{dy}{dt} = y + t, t > 0$.

2(**18 pts**) Solve the initial value problem to the equation $\dfrac{dy}{dt} - ty^2 = t,\ y(0) = 1$.

3(**18 pts**) (a) Sketch the phase line of the equation $\dfrac{dy}{dt} = ay - by^2$, where a, b are positive constants.
(b) Classify the stability of each equilibrium solution.
(c) Sketch a solution portrait of the equation in the ty-plane. Find the limit $\lim_{t\to\infty} y(t)$ if $y(0) = \frac{a}{2b}$.

4(**18 pts**) Use the method of undetermined coefficients to find a suitable **form** for a particular solution $y_p(t)$ to the equation $y'' - 4y' - 5y = 1 + 5\sin 2t + te^{5t}$. (Do not solve for the coefficients! Also you need to segregate the nonhomogeneous term into the sum of smaller terms to which the method applies.)

5(**18 pts**) The characteristic equation for a homogeneous, 6th order, linear equation $a_6y^{(6)} + a_5y^{(5)} + a_4y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0$ with constant coefficients is factored as $a_6r^6 + a_5r^5 + a_4r^4 + a_3r^3 + a_2r^2 + a_1r + a_0 = 5(r+5)^2(r^2+2r+5)^2 = 0$. Find the general solution to the differential equation.

6(**18 pts**) Find a particular solution to the equation $(\sin t)y'' + b(t)y' + c(t)y = \cos t$ if $y_1(t) = 1, y_2(t) = \sin t$ are two solutions to the homogeneous equation $(\sin t)y'' + b(t)y' + c(t)y = 0$. (Formula you need: $\int \csc t dt = -\ln|\csc t + \cot t| + C$.)

7(**18 pts**) (a) Find the general to the system of equations:

$$\begin{cases} x' = 4x - 6y \\ y' = 3x - 5y \end{cases}$$

(b) Sketch a phase portrait of the system and classify the stability and the type of the equilibrium point $(0, 0)$.
(c) Find the solution that satisfies $x(0) = 0, y(0) = 1$.

8(**18 pts**) Consider the system of equations

$$\begin{cases} x' = x(1 - x + y) \\ y' = y(1 - 2y + x). \end{cases}$$

(a) Find all equilibrium solutions of the system.
(b) Sketch a phase portrait of the system in the first quadrant (which must include typical vector fields on and off the nullclines and a few solution curves).

9(**18 pts**) Consider the same system of equations of # 8 above:

$$\begin{cases} x' = x(1 - x + y) \\ y' = y(1 - 2y + x). \end{cases}$$

(a) Find the linearized equation at the equilibrium point that is not on the axes.
(b) Find the eigenvalues of the linearized equations.

10(**18pts**) Find the Laplace Transform $\mathcal{L}\{y(t)\}(s)$ of the solution to the initial value problem

$$3y''' + 2y'' + y' = 2t^3, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2.$$

(Do Not Solve for $y(t)$!)

11(**20 pts**) Use the method of Laplace Transform to solve the initial value problem

$$y'' - 3y' + 2y = 2, \quad y(0) = 1, \quad y'(0) = 3.$$

Bonus 3pts: The name of the pirate ship that Captain Jack Sparrow tried to get back is: $\underline{TheWhiteShark}$, $\underline{TheDuntless}$, $\underline{TheInterceptor}$, $\underline{TheBlackPearl}$.

Formulas that you need:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}(t) = \frac{e^{at} - e^{bt}}{a-b}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-a)(s-b)}\right\}(t) = \frac{ae^{at} - be^{bt}}{a-b}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)(s-c)}\right\}(t) = \frac{1}{(a-b)(a-c)}e^{at} + \frac{1}{(b-c)(b-a)}e^{bt} + \frac{1}{(c-a)(c-b)}e^{ct}$$

The End