Chapter Two

Differential Equations

2.1 LINEAR EQUATIONS

We consider ordinary differential equations of the following form

$$\frac{dx}{dt} = f(t, x).$$

where x is either a real variable or a vector of several variables, t is a real, independent variable which is often conveniently thought as a time, and f is a given function of both variables. The **dimension** of the equation is referred to the dimension of the dependent variable x.

In some cases, f does not depend on both variables. If f does not depend on the time variable, the equation

$$x' = f(x)$$

is called an **autonomous equation**.

For simplicity we denote the derivative either by 'prime' $x' = dx/dt$ or by 'dot' $\dot{x} = dx/dt$. Thus the equation can also be written as

$$x' = f(t, x) \text{ or } \dot{x} = f(t, x).$$

One Dimensional Equations

We begin by considering one of the simplest types of equations – 1 dimensional, linear, with constant coefficient

$$x' = f(x) = ax, \tag{2.1}$$

where a is a constant.

A solution of the equation is a function $x(t)$ of time t so that the equation holds when $x(t)$ is substituted in. For example, $x(t) = 3e^{2t}$ is a solution to the equation

$$x' = 2x$$

because by the chain rule and the derivative of exponential functions,

$$x'(t) = 3e^{2t} \cdot 2 = 2x(t).$$

The equation thus holds and $3e^{2t}$ is a solution. In contrast, $x(t) = \sin t$ is not a solution because

$$x'(t) = \cos t \neq 2x(t) = 2\sin(t).$$

Method of Trial Solutions. Finding solutions to an arbitrarily given equation is never trivial. Different equation types require different approaches, which usually are not transferable from one type to another. Fortunately for the equation (2.1) under consideration, what takes to solve it is an educated insight not beyond the realm of elementary functions.

The equation $x' = ax$ says that a solution $x(t)$ is such a function that its derivative $x'(t)$ is essentially itself $\approx x(t)$. What types of elementary functions fit this description? There are many. But a simple candidate is the family of exponential functions of the following form

$$Ce^{rt}$$

with C, r being arbitrary parameters. The question then becomes for what parameter values of C and r is Ce^{rt} a solution? To this extent, the equation $x' = ax$ is the only constraint and there is only one way to find it out – plug it in and see if it fits. First,

$$[Ce^{rt}]' = Ce^{rt} \cdot r.$$

Setting it to the right hand side of the equation,

$$[Ce^{rt}]' = Ce^{rt} \cdot r = a[Ce^{rt}]$$

which is true if $r = a$. Hence, we conclude that

$$x(t) = Ce^{at}$$

is a solution to the equation for any arbitrary constant C.

General Solutions: The appearance of one arbitrary constant in a solution is expected for any first order differential equation of one variable because of the following reasons. To solve differential equations is to find antiderivatives, almost always indirectly and implicitly, and each differentiation of one variable generates one arbitrary 'integration' constant upon the completion of such an antiderivative process. Thus, as a rule of thumb, we anticipate two arbitrary constants for the solutions of a second order equation of one variable, or a first order equation of two variables, and so on. In general, we anticipate $n \times m$ many arbitrary constants for an nth order equations of m many variables. Such a solution containing the right amount of arbitrary constants is called a **general solution**. Hence, $x(t) = Ce^{at}$ is a general solution to the equation $x' = ax$.

The arbitrary constant C can be determined by specifying one additional condition to the equation. The case we consider almost exclusively is the **initial condition** which in general takes the form

$$x(t_0) = x_0$$

with t_0, x_0 prescribed. In most cases we take $t_0 = 0$. For example, using the initial condition

$$x(0) = x_0$$

the arbitrary constant C is determined uniquely by the following

$$x_0 = x(0) = Ce^{a0} = C.$$

Hence, $x(t) = x_0 e^{at}$ is *the* solution to the **initial value problem**

$$\begin{cases} x' = f(x) = ax \\ x(0) = x_0 \end{cases}$$

Example 2.1.1 Solve the initial value problem

$$\begin{cases} x' = -2x \\ x(1) = 3 \end{cases}$$

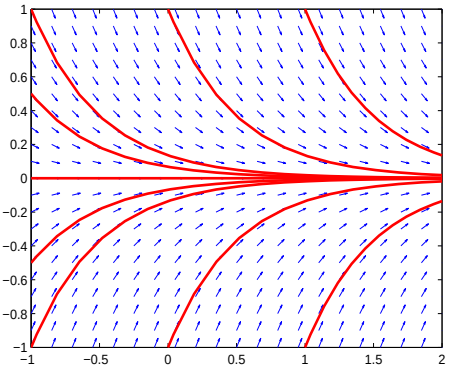


Figure 2.1 Solution portrait of Example 1.

Solution: Since $a = -2$, $x(t) = Ce^{-2t}$ is the general solution. By the initial condition

$$3 = x(1) = Ce^{-2\cdot 1} \Longrightarrow C = 3e^2$$

and the solution to the initial value problem is

$$x(t) = 3e^2 e^{-2t} = 3e^{2(1-t)}.$$

◎

Typical solutions of the example are sketched in Fig.2.1, as well as its vector field. Such a portrait is called a **solution portrait**.

Important properties of this type of equations are the following

- $x(t) \equiv 0$ is a constant solution, called an equilibrium solution.
- If $a < 0$, all solutions exponentially converge to the equilibrium solution as t tends to infinity

  $$\lim_{t\to\infty} Ce^{at} = 0.$$

  The equilibrium solution $x = 0$ is said to be **asymptotically stable**.
- If $a > 0$, all solutions, except for the equilibrium solution, exponentially diverge to infinity as t tends to infinity

  $$\lim_{t\to\infty} Ce^{at} = \begin{cases} \infty, & \text{if } C > 0 \\ -\infty, & \text{if } C < 0 \end{cases}$$

  Such an equilibrium solution $x = 0$ is said to be **asymptotically unstable**.

Two Dimensional Equations

Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

be a vector of two real variables x_1, x_2, and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a 2×2 matrix. The system of two linear equations of the following form

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{cases}$$

can also be written in its matrix form

$$x' = Ax.$$

Method of Trial Solutions. Similar to the rationale for the scalar case, we try it out to see if exponential functions of the following form

$$x = \xi e^{\lambda t} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^{\lambda t}$$

are solutions. Using the derivative convention

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

and its resulting algebraic rules, we have

$$(\xi e^{\lambda t})' = \xi e^{\lambda t} \cdot \lambda.$$

Setting it to the right hand side of the equation, we have

$$(\xi e^{\lambda t})' = \xi e^{\lambda t} \cdot \lambda = A(\xi e^{\lambda t}) = (A\xi)e^{\lambda t}.$$

Cancelling out $e^{\lambda t}$ we derive the following relation

$$A\xi = \lambda\xi \iff (A - \lambda I)\xi = 0.$$

Since we seek solutions other than the trivial equilibrium solution $x = 0$, we want $\xi \neq 0$ for the solution candidate $\xi e^{\lambda t}$. Hence the condition above implies that λ must be an eigenvalue of the matrix A and ξ must be an eigenvector of the eigenvalue λ. By a reason of matrix theory, λ is an eigenvalue if and only if it is the root the characteristic equation

$$|A - \lambda I| = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

Depending on the nature of the roots, the method follows up one of three possibilities, each goes through a distinct set of steps.

Case I: Distinct Real Eigenvalues.

In this case, both λ_1, λ_2 are real and unequal. We first find one eigenvector ξ_1 for eigenvalue λ_1 by solving the equation

$$(A - \lambda_1 I)\xi_1 = \begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives one **eigensolution**

$$\mathbf{x}_1(t) = \xi_1 e^{\lambda_1 t}$$

Repeat the same steps for eigenvalue λ_2 to find an eigenvector ξ_2 and an eigensolution

$$\mathbf{x}_2(t) = \xi_2 e^{\lambda_2 t}$$

By the Superposition Principle, a general solution is obtained as a linear combination of the eigensolutions:

$$x(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1\xi_1 e^{\lambda_1 t} + C_2\xi_2 e^{\lambda_2 t} \tag{2.2}$$

where C_1, C_2 are two arbitrary constants.

Example 2.1.2 Solve the initial value problem

$$x' = \begin{bmatrix} -5 & 2 \\ -4 & 1 \end{bmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution: Solve the characteristic equation

$$|A - \lambda I| = \lambda^2 + 4\lambda + 3 = 0.$$

We get $\lambda_1 = -1, \lambda_2 = -3$. For λ_1, solve the eigenvector equation

$$(A - \lambda_1 I)\xi = \begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It reduces to

$$-4u_1 + 2u_2 = 0 \iff u_2 = 2u_1.$$

Since only one nonzero eigenvector is needed, we pick one in a practical way as simple and as convenient as possible. Take $u_2 = 1$ and thus $u_2 = 2$ follows. So the corresponding eigensolution is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$

Repeat the same steps for $\lambda_2 = -3$ we find a second eigensolution

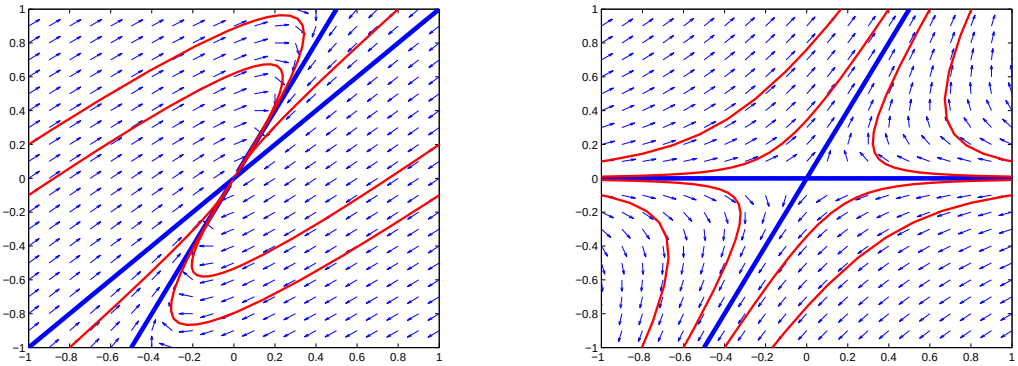
$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.$$

The general solution then is

$$x(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$$

The solution to the initial value problem is picked out from the general solution by fixing the arbitrary constants C_1, C_2. To this end, we use the initial condition

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = x(0) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$



(a) (b)

Figure 2.2 Phase portraits. (a) Example 2, (b) Example 3.

This is a linear equation for C_1, C_2. Solve it to get $C_1 = -2, C_2 = 3$, and the solution is

$$x(t) = -2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} -2e^{-t} + 3e^{-3t} \\ -4e^{-t} + 3e^{-3t} \end{pmatrix}$$

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Fig.2.2(a) is a **phase portrait** of the equations in the phase plane of x_1, x_2. It contains the eigensolutions, and a few typical solutions. Typical features are the following:

- Eigensolutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ lie on two radial lines through the eigenvectors.
- Because the eigenvalues are all negative, all solutions converge to the trivial solution $x = 0$, i.e. $\lim_{t\to\infty} x(t) = 0$.
- Because $\lambda_2 < \lambda_1 < 0$, $e^{\lambda_2 t} = e^{-3t}$ decays to 0 faster than $e^{\lambda_1 t} = e^{-t}$ does, all solutions are dominated by the slower decaying term $C_1\xi_1 e^{-t}$:

  $$x(t) \sim C_1\xi_1 e^{\lambda_1 t}, \text{ as } t \text{ tends to } \infty.$$

  This explains why all solutions are asymptotically tangent to the radial line through the eigenvector ξ_1.

The equilibrium point with all negative eigenvalues is called a **sink**. If the eigenvalues are all positive, all solutions other than the equilibrium point diverge without bound. Such an equilibrium point is called a **source**.

Example 2.1.3 Find a general solution to the equation and sketch a phase portrait.

$$x' = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} x$$

Solution: Solving the characteristic equation

$$|A - \lambda I| = \lambda^2 - 2\lambda - 6 = 0$$

we get $\lambda_1 = -2, \lambda_2 = 4$. Solving the eigenvector equation $(A - \lambda_i I)\xi = 0$, we have

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

for λ_1 and λ_2 respectively. The corresponding general solution is

$$x(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}$$

Fig.2.2(b) is a phase portrait of the equation. It is important to note the following typical features:

- The eigensolutions $\mathbf{x}_i(t)$ lie on two lines determined by the eigenvectors. $\mathbf{x}_1(t)$ converges to the origin, it is called a stable eigensolution. $\mathbf{x}_2(t)$ flies off to infinity, it is called an unstable eigensolution.
- In forward time $t \to \infty$, a non-eigensolution $x(t)$ is dominated by the unstable eigensolution

  $$x(t) \sim C_2\xi_2 e^{4t},$$

  asymptotically parallel with the unstable eigenvector ξ_2.
- In backward time $t \to -\infty$, $x(t)$ is dominated by the stable eigensolution

  $$x(t) \sim C_1\xi_1 e^{-2t},$$

  asymptotically parallel with the stable eigenvector ξ_1.

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As a result, all solutions, except for the equilibrium solution $x = 0$ and the stable eigensolutions $C_1\mathbf{x}_1(t)$, diverge without bound. Such an equilibrium point with eigenvalues of opposite signs is called a **saddle**. ◎

One more subcase to consider is when one of the eigenvalues is zero, say $\lambda_1 = 0$. In this case, the general solution is also given by Eq.(2.2)

$$x(t) = C_1\xi_1 + C_2\xi_2 e^{\lambda_2 t}$$

with C_1, C_2 being arbitrary constants. For $C_2 = 0$, each of the point $x(t) \equiv C_1\xi_1$ is an equilibrium solution including the trivial one with $C_1 = 0$. The trivial equilibrium point is referred to as a **stable node** if $\lambda_2 < 0$ and a **unstable node** if $\lambda_2 > 0$.

Case II: Repeat Eigenvalues.

In this case, $\lambda_1 = \lambda_2 = \lambda$ which must be a real. It may has two eigenvectors ξ_1, ξ_2 which are not constant multiple of each other, i.e., $\xi_1 \neq C\xi_2$ for any C, or it has only one eigenvector ξ and all other eigenvectors are constant multiple of ξ.

If it is the first scenario, it follows the same procedures as in the distinct real eigenvalue case to obtain a general solution

$$x(t) = C_1\xi_1 e^{\lambda t} + C_2\xi_2 e^{\lambda t}.$$

Generalized Eigensolutions. It is the second scenario that a refined approach is required. The method of trial solutions so far gives us only one eigensolution $\mathbf{x}_1(t) = \xi e^{\lambda t}$. It is not enough to form a general solution which requires two arbitrary constants. The goal is to find a second solution that is not a constant multiple of $\mathbf{x}_1(t)$.

To do this, we follow the same rationale that led to our first eigensolution $\mathbf{x}_1(t)$, which is obtained from the pool of candidate functions of the form $\xi e^{\lambda t}$ whose derivatives are in the same pool. To find another solution outside the pool, we enlarge it but require still that its functions fit the characterization that their derivatives are also in the same pool. Here is such an enlarged pool: functions of this form

$$(\eta + t\xi)e^{\lambda t},$$

which are products of first degree polynomials and exponentials of a fixed parameter λ.

A few simple algebraic manipulations lead to the following

$$[(\eta + t\xi)e^{\lambda t}]' = (\lambda\eta + \xi + t\lambda\xi)e^{\lambda t} = A(\eta + t\xi)e^{\lambda t} = (A\eta + tA\xi)e^{\lambda t}$$

Cancelling out $e^{\lambda t}$ and equating terms with t and terms without t (since we only need to find a solution as pragmatically as possible), we get the following conditions on ξ, η, λ:

$$\begin{array}{rcl} A\xi &=& \lambda\xi \\ A\eta &=& \lambda\eta + \xi \end{array} \iff \begin{array}{rcl} (A - \lambda I)\xi &=& 0 \\ (A - \lambda I)\eta &=& \xi \end{array}$$

The first equation is the same eigenvalue-eigenvector relation obtained when the functional pool is the original smaller one, $\xi e^{\lambda t}$. Solution η to the second equation is called a **generalized eigenvector**. To find one, we first solve the eigenvalue-eigenvector problem to get λ, ξ, and then solve the equation

$$(A - \lambda I)\eta = \xi$$

for the generalized eigenvector, if indeed λ is a double eigenvalue, ξ is the only eigenvector of λ and all others are constant multiple of ξ.

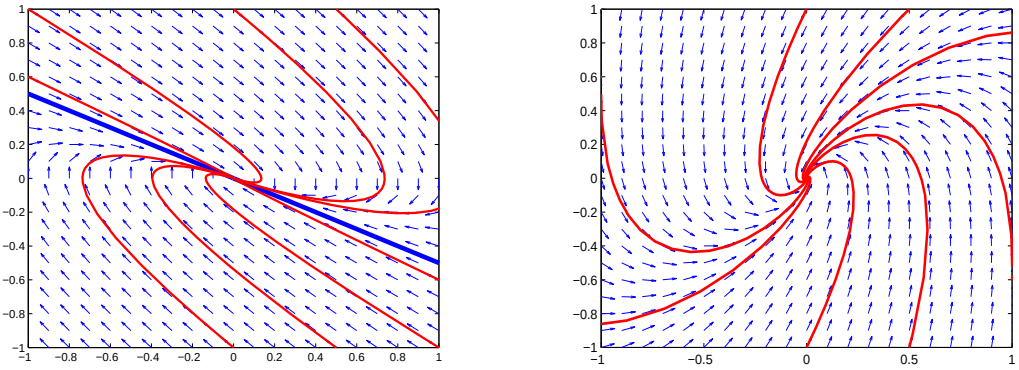
The resulting solution

$$(\eta + t\xi)e^{\lambda t}$$

to the differential equation is called a **generalized eigensolution**.

Example 2.1.4 Find a general solution to the equations and sketch a phase portrait,

$$x' = \begin{bmatrix} 0 & 4 \\ -1 & -4 \end{bmatrix} x$$



(a) (b)

Figure 2.3 Phase portraits. (a) Example 4, (b) Example 5.

Solution: Solving the characteristic equation

$$|A - \lambda I| = \lambda^2 + 4\lambda + 4 = 0$$

we get $\lambda_1 = \lambda_2 = \lambda = -2$. Solve the eigenvector equation

$$(A - \lambda I)\xi = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get

$$\xi = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ and the eigensolution } \mathbf{x}_1(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-2t}.$$

To find a second solution, we solve the generalized eigenvector equation

$$(A - \lambda I)\eta = \xi \implies \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

It reduces to

$$v_1 + 2v_2 = -1.$$

Since we seek just one solution, assigning $v_2 = 0, v_1 = -1$ will do. Hence, a generalized eigenvector and the corresponding generalized eigensolution are

$$\eta = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{x}_2(t) = (\eta + t\xi)e^{\lambda t} = \begin{pmatrix} -1 - 2t \\ t \end{pmatrix} e^{-2t} \text{ respectively.}$$

A general solution is thus given as

$$x(t) = C_1\xi e^{-2t} + C_2(\eta + t\xi)e^{-2t} = \begin{pmatrix} -2C_1 + C_2(-1 - 2t) \\ C_1 + C_2 t \end{pmatrix} e^{-2t}.$$

Fig.2.3(a) is a solution portrait of the equations in the phase plane of x_1, x_2. It is important to note the following properties:

- There is one radial line that contains solutions, the eigenvector line through the origin and the eigensolutions.
- Because of the negative eigenvalue, all solutions converge to the trivial solution $x = 0$, i.e. $\lim_{t\to\infty} x(t) = 0$.
- In forward time as $t \to \infty$, the term $t\xi e^{-2t}$ decays slower than the terms without the multiple of t, it approximates and dominates the solution
$$x(t) \sim C_2 \xi t e^{\lambda_1 t}.$$
This explains why all solutions are asymptotically tangent to the eigenvector radial line through the origin.
- In backward time as $t \to -\infty$, the term $t\xi e^{-2t}$ diverge faster than the terms without the multiple of t, it also approximates and dominates the solution
$$x(t) \sim C_2 \xi t e^{\lambda_1 t}.$$
This explains why all solutions are asymptotically parallel to the eigenvector line.

Remark. Because of the last two properties, a phase portrait for such a case can be drawn without finding the generalized eigensolution. All needed are the eigensolution and the vector field at one point. For example, at point $(0, 1)$ the vector field is
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & -4 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix},$$
pointing right-down. The only way to incorporate the asymptotic behaviors and this piece of information on vector field is to have the phase portrait as shown.

Case III: Complex Eigenvalues.

Let $\lambda_{1,2} = \alpha \pm i\beta$ be the eigenvalues. In this case, we only need to use one eigenvalue, say $\lambda_1 = \alpha + i\beta$. Like all other cases, we proceed to find an eigenvector, which is usually a vector of complex entries. Let
$$\xi = \begin{pmatrix} u_1 + iv_1 \\ u_2 + iv_2 \end{pmatrix}$$
be an eigenvector with u_1, u_2, v_1, v_2 real numbers. We then separate it into real and imaginary parts in vector form as
$$\xi = u + iv \text{ where } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$
That is both u and v are vectors of real entries. Following the deduction for solutions, we know $z(t) = (u + iv)e^{\lambda t} = (u + iv)e^{(\alpha+i\beta)t}$ is a solution. However, it is complex valued, which is not what we eventually want. To find two real valued functions, we further separate $z(t)$ into real and imaginary

parts of the form $z(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$ with both $\mathbf{x}_1(t), \mathbf{x}_2(t)$ real valued functions. (See example below to learn how to carry out the separation.) Since $z(t)$ is a solution, the following lines of reasoning show that both $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are solutions as well.

$$\mathbf{x}_1'(t) + i\mathbf{x}_2'(t) = z'(t) = Az(t) = A\mathbf{x}_1(t) + iA\mathbf{x}_2(t).$$

Equating real and imaginary parts, we have

$$\mathbf{x}_1'(t) = A\mathbf{x}_1(t) \text{ and } \mathbf{x}_2'(t) = A\mathbf{x}_2(t).$$

Hence, both $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are real valued solutions, and the corresponding general solution is given as usual

$$x(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t).$$

Example 2.1.5 Find a general solution to the equations and sketch a phase portrait.

$$x' = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix} x$$

Solution: Solving the characteristic equation

$$|A - \lambda I| = \lambda^2 + 6\lambda + 13 = 0$$

gives $\lambda_{1,2} = -3 \pm i2$. The eigenvector equation for λ_1 is

$$(A - \lambda_1 I)\xi = \begin{bmatrix} 2 - i2 & -2 \\ 4 & -2 - i2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which in components are

$$\begin{aligned} (2 - i2)z_1 - 2z_2 &= 0 \\ 4z_1 - (2 + i2)z_2 &= 0. \end{aligned}$$

The two equations are redundant of each other, differing only by a constant multiple (e.g., multiplying $1 + i$ and the first equation becomes the second equation). A simple eigenvector is obtained by assigning $z_1 = 1$ which gives $z_2 = 1 - i$. Hence

$$\xi = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} \text{ and } z(t) = \begin{pmatrix} 1 \\ 1 - i \end{pmatrix} e^{(-3+i2)t}.$$

To separate $z(t)$ into its real and imaginary parts, we use the **Euler formula** for complex exponential:

$$\boxed{e^{a+ib} = e^a(\cos b + i \sin b).}$$

Use it to get

$$e^{(-3+i2)t} = e^{-3t+i2t} = e^{-3t}(\cos(2t) + i \sin(2t)).$$

Now using the complex multiplication rule

$$(a + ib)(c + id) = ac - bd + i(ad + bc),$$

we separate the solution $z(t)$ into its real and imaginary parts as follows:

$$\begin{aligned} z(t) &= \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{-3t+i2t} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{-3t}(\cos(2t) + i\sin(2t)) \\ &= \begin{pmatrix} e^{-3t}\cos(2t) + ie^{-3t}\sin(2t) \\ e^{-3t}(\cos(2t)+\sin(2t)) + ie^{-3t}(\sin(2t)-\cos(2t)) \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix} + ie^{-3t}\begin{pmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix} \\ &:= \mathbf{x}_1(t) + i\mathbf{x}_2(t). \end{aligned}$$

The corresponding general solution is

$$\begin{aligned} x(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1e^{-3t}\begin{pmatrix} \cos(2t) \\ \cos(2t)+\sin(2t) \end{pmatrix} + C_2e^{-3t}\begin{pmatrix} \sin(2t) \\ \sin(2t)-\cos(2t) \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} C_1\cos(2t) + C_2\sin(2t) \\ (C_1-C_2)\cos(2t) + (C_1+C_2)\sin(2t) \end{pmatrix} \end{aligned}$$

Fig.2.3(b) is a solution portrait of the equations in the phase plane of x_1, x_2. Important features are as follows:

- Because of the negative real part of the eigenvalue, all solutions converge to the trivial solution $x = 0$, i.e. $\lim_{t\to\infty} x(t) = 0$.
- All solutions spiral around the equilibrium solution $x = 0$. The direction of spiral can be determined by the vector field at one point. For example, at point $(1, 0)$, the vector field is

  $$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} -1 & -2 \\ 4 & -5 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$

  It points left-up. Therefore the spiral is counterclockwise.

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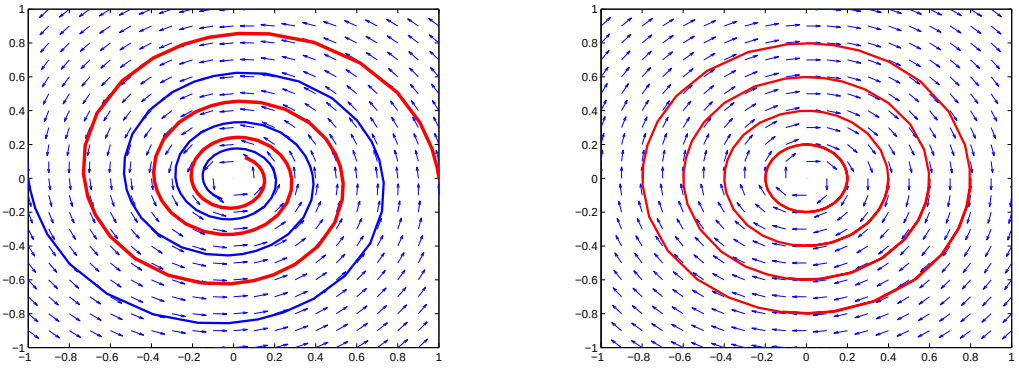
Remark. Using the information of the real part of the complex eigenvalues and the vector field at one point, one can qualitatively sketch the phase portrait of the equation without solving it. For example, had the real part of the eigenvalues of the example above been positive and the vector field at $(1, 0)$ been the same, solutions would spiral counterclockwise and away from the origin.

◎

Fig. 2.4 illustrates two more examples of the complex eigenvalue case. Fig.2.4(a) is for the coefficient matrix $A = \begin{bmatrix} -0.1 & -1 \\ 1 & -0.1 \end{bmatrix}$ for which the real part, -0.1, of the eigenvalues is relatively small in magnitude, resulting in a looser spiral than that of Example 2.1.5 seen in Fig.2.3(b). Fig.2.4(b) is for the coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for which the real part is zero, giving rise to all closed cycles for its orbits. The trivial equilibrium point $(0, 0)$ is referred to as a **center**.

◎



(a) (b)

Figure 2.4 Phase portraits with complex engensolutions: (a) A looser and clockwise spiral; (b) A center.

The trivial equilibrium solution $\mathbf{x} = 0$ to the linear equation $\mathbf{x}' = A\mathbf{x}$ is said to be **asymptotically stable** $\lim_{t\to+\infty} \mathbf{x}(t) = 0$. It is not hard to show that $\mathbf{x} = 0$ is asymptotically stable for a linear system if and only if all eigenvalues of A have a negative real part. On the other hand, if one eigenvalue has a positive real part, then the trivial equilibrium point is **asymptotically unstable**. For the case of Fig.2.4(b) for which the real part of the complex eigenvalues of A vanishes, the equilibrium point is **stable** but not asymptotically stable.

Higher Dimensional Systems and the Routh-Hurwitz Criteria.

Theorem 2.1 (Routh-Hurwitz Criteria[1]) *For the characteristic equation of an $n \times n$ coefficient matrix A of a linear system of equations $\boldsymbol{x}' = A\boldsymbol{x}$,*

$$|\lambda I - A| = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0,$$

the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \ldots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \ldots & b_k \end{vmatrix}$$

and $b_i = 0$ if $i \geq n$.

For the specific case of $n = 2$, the criteria reduces to

$$b_1 = -\mathrm{tr}(A) > 0 \text{ and } b_2 = \det(A) > 0.$$

For $n = 3$, it becomes

$$b_1 = -\mathrm{tr}(A) > 0, b_2 = \sum_{k=1}^{3} \det(A_k) > 0, b_3 = -\det(A) > 0, \text{ and } b_1 b_2 > b_3,$$

where $\mathrm{tr}(A)$ is the trace of A (the sum of A's diagonal entries), and A_k is the matrix obtained from A by deleting the kth row and the kth column of A.

Exercises 2.1

1. Find the solutions to the equations

   (i) $x' = -2x,\ x(0) = 3$

   (ii) $x' = \dfrac{x}{2},\ x(1) = 2$

2. Find general solutions of the equations, determine the stability of the equilibrium solution $x = 0$, and sketch the phase portraits, which should include the eigensolutions of the equations if apply.

   (i) $x' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} x$

   (ii) $x' = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x$

   (iii) $x' = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} x$

   (Eigenvalues: (i) $-1, -2$. (ii) $2, -1$. (iii) $2 \pm i$.)

3. Determine the stability of the equilibrium point $x = 0$ of the equations.

   (i) $x' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} x$

   (ii) $x' = \begin{bmatrix} 1 & -3 \\ 6 & -5 \end{bmatrix} x$

2.2 PHASE LINE AND BIFURCATION

Having solutions in formulas to differential equations are rare nor always desirable. In most cases we seek a qualitative understanding of the solution structure without displaying the solutions in formulas. The nature of such methods are inevitably conceptual and geometrical.

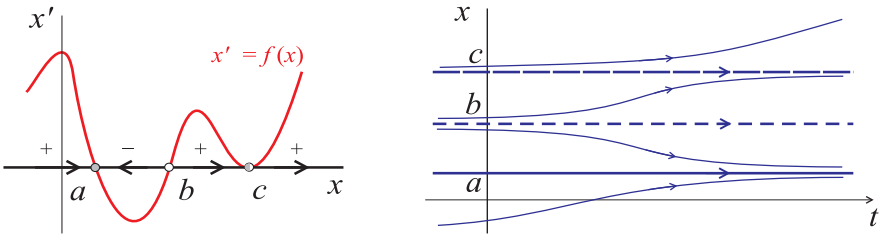
Here we introduce a qualitative method for the simplest class of nonlinear differential equations: the autonomous equations of one dimension:

$$x' = f(x).$$

A solution $x(t)$ is simply a function of the time variable t. They are remarkably simple for equations. The solution $x(t)$ can behave only in one of the following three ways:

- $x(t)$ remains constant for all t, which is called an **equilibrium solution**, or **equilibrium point**.
- $x(t)$ monotone increases either without bound or approaches an equilibrium point.
- $x(t)$ monotone decreases either without bound or approaches an equilibrium point.

These properties are captured by the graphs below.



On the left is the graph of the right hand of the equation. Its x-intercepts, a, b, c, captures all the equilibrium points. These points divide the phase space, i.e., the x-axis, into intervals in each of which the rate of change function f can take on one sign, positive or negative, since the only places where it can possibly change signs are the equilibrium points at the end points of the intervals. This implies that starting at any point inside an interval with positive f, the solution increases in time and stays in the same interval. Similarly, a solution decreases and stays in an interval of negative sign of f if it starts in the same interval. The arrows designate the directions of the orbital monotonicity.

On the right is a qualitative depiction of some typical solutions as functions of the independent variable t. Equilibrium solutions stay as horizontal lines. Others either converge to or away from equilibrium solutions. It is useful to note that this solution portrait does not offer substantially additional information than the features already annotated on the x-axis in the

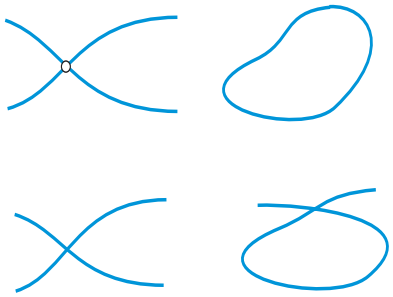
Existence and Uniqueness of Solutions. An important theoretical question is if an initial value problem

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a solution and if it has a unique solution. The answer depends on the function f. If f is continuously differentiable with respect to both t and x in a neighborhood of the initial point (t_0, x_0), then it has a unique solution. In fact, the same conclusion holds for somewhat weaker conditions, such as f is Lyapunov continuous. (A typical example of Lyapunov continuity is the function of absolute value $f(t,x) = |x|$. Piecewise linear functions is another.) Hence, the existence and uniqueness question is rarely an issue for differential equation models from sciences and engineering fields.

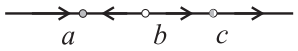
One immediate and importance consequence to the uniqueness of solutions to initial value problems is that solutions having different initial conditions never intersect at any moment in time. For example, the solution plot of two different solutions in the tx-space must not intersect.

A special consequence is worth noticing for autonomous equation $x' = f(x)$ in any dimensions. It is based on the property specific for such equations that if $x(t)$ is a solution, so is any time translation $x(t+\theta)$. (Just differentiate and plug it into the equation to check.) The projection of a solution $(t, x(t))$ to the **phase space** x is the parameterized curve $x(t)$, which is called an **orbit**. The uniqueness property of solutions to initial value problems implies that an orbit cannot *cross* itself nor another orbit. For examples, the top two figures on the right are possible orbits but the bottom two are not. (There are five orbits depicted in the upper-left figure, including the open dot representing an equilibrium orbit.)



One important exception is Newton's law of gravity which introduces singularities to equations governing celestial bodies. As a result solutions may not be unique coming out the singularities, and celestial bodies do smash onto each others from time to time.

left figure. In fact, the diagram has captured all important qualitative information of the equation. For this reason we refer to such a plot as the **phase line**, which is extracted and depicted alone as below.

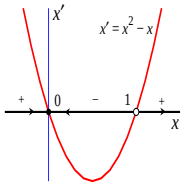


You may have notices that solutions do not behave the same near all equilibrium points. We have the following classifications:

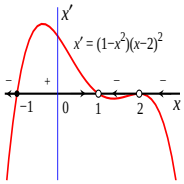
- An equilibrium point is a **sink** if all solutions nearby converge to it.
- An equilibrium point is a **source** if all solutions nearby diverge from it.
- An equilibrium point is a **node** if some solutions converge to it and some others diverge from it.

In the example depicted above, a is a sink, b is a source, and c is node.

We also say that a sink is **asymptotically stable** or a sink attracts all solutions nearby, and a source is **asymptotically unstable** or it repels all solutions nearby. A node is simply **unstable**.



(a)



(b)

Example 2.2.1 Sketch the phase lines of equations and classify the equilibrium points.

$$\text{(a)}\ \ x' = x^2 - x \qquad\qquad \text{(b)}\ \ x' = (1-x^2)(x-2)^2$$

Solution: (a) First we plot the right hand of the equation as shown. It intersects the x-axis at $x = 0, 1$, which are the equilibrium points. The signs of $f(x) = x^2 - x$ in intervals partitioned by the equilibrium points are determined by the relative position of the graph to the x-axis, $+$ if it is above the x-axis, $-$ if it is below. The arrows are placed according to the signs, right arrow for $+$ and left arrow for $-$. These intervals with assigned arrows are called **orbits**, which are the projections to the **phase space** x of the solutions $(t, x(t))$ from the tx-plane. We conclude from the phase line plot that equilibrium point $x = 0$ is a sink and $x = 1$ is a source.

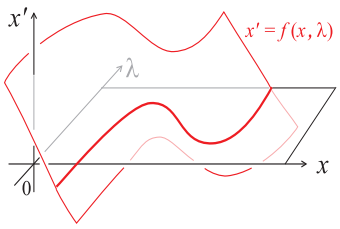
(b) Following the same steps, the solution is given by the graph. For stabilities of equilibrium points, $x = -1$ is a sink, $x = 1$ is a source, and $x = 2$ is a node.

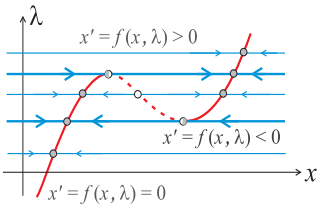
Phase Lines with Parameter and Bifurcations. We consider next equations that depends on one parameter,

$$x' = f(x, \lambda),$$

where t, x are the variables and λ is the parameter. For each fixed λ value, the qualitative dynamics of the equation is captured by its phase line. The structure of the phase line may change for a different λ value. To see such changes, we may plot the phase lines one λ value a time. But this is cumbersome and inefficient — it is not possible to draw phase lines for all λ values.

There is a more effective way. It is based on a lesson learned from the phase line plot. That is, the equilibrium points of the equation together with the sign of the right hand of the equation determine the phase line structure. The **phase line method with parameter** is illustrated below.





The steps are summarized as follows.

1. In the $x\lambda$-plane, plot the level curve $f(x, \lambda) = 0$, called the **equilibrium branch**. A point, (x, λ), on the curve gives an equilibrium point x of the equation for the given parameter λ.
2. The equilibrium branch or branches divides the region of interest into subregions, in each of which $f(x, \lambda)$ does not vanish and therefore it has a fixed sign.
3. To sketch the phase line for a fixed value λ, just draw the horizontal line through λ. Any intersection of the line with the equilibrium brand $f(x, \lambda) = 0$ is an equilibrium point for the equation for that λ value. The line is partitioned into intervals by the equilibrium points, falling into $f > 0$ and $f < 0$ regions. The orbital structure is then drawn on the line as you would for a phase line.
4. Use solid curves for the parts of the equilibrium branch whose equilibrium points are stable and dash curves for the ones whose equilibrium points are unstable. They are referred to as the **stable equilibrium branches** and **unstable equilibrium branches**, respectively.

In the hypothetical illustration above, we see that there are typical parameter values for which the qualitative structure of phase lines do not change if one change the parameter near by, such as the ones not going through the critical points of the equilibrium branch. Yet there are a few atypical parameter values that stand out, such as the ones going through the critical points. A slight increase or decrease from these special values give rise to qualitatively different phase lines, especially in the number of equilibrium points and their stabilities. We refer to such a phenomenon as **bifurcation**. In particular, we call a point, $(\bar{x}, \bar{\lambda})$, a **bifurcation point**, if $f(\bar{x}, \bar{\lambda}) = 0$ and if the phase line changes qualitatively near the equilibrium point $\bar{x}$ when λ varies slightly from $\bar{\lambda}$.

We will introduce the types of bifurcation to be encountered in this chapter through examples. These are *saddle-node bifurcation*, *transcritical bifurcation*, and *Hopf bifurcation*. Other types that will not be used are introduced through exercises. The Hopf bifurcation is for 2-dimensional systems and it will be introduced in a later section.

Saddle-Node Bifurcation.

Example 2.2.2 Consider the logistic model with a constant harvest rate H,

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H \tag{2.3}$$

where r is the maximum per-capita growth rate, K is the carrying capacity, and H is the constant harvest rate. Sketch the phase lines using H as the parameter.

Solution: The equilibrium branch is defined by $rP(1 - \frac{P}{K}) - H = 0$. By chance, it can be solved as $H = rP(1 - \frac{P}{K})$ in the PH-plane. The branch divides the plane into two parts: $P' > 0$ below the branch and $P' < 0$

Logistic Model. Let $P(t)$ be a population measure of a species at time t. The time t can be in second, or day, or year, etc, and population P can be in a nonnegative number for the total biomass, or density. The derivative $\dfrac{dP(t)}{dt}$ is the growth rate, and

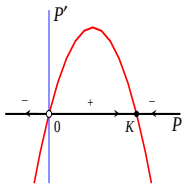
$$\frac{1}{P(t)}\frac{dP(t)}{dt}$$

defines the **per-capita growth rate**. $P'(t) > 0$, the population increases. $P'(t) < 0$, it decreases. $P'(t) \equiv 0$ for all t, it stays at an equilibrium, or steady state, $P(t) \equiv$ a constant.

The logistic model for population growth is to assume that the per-capita growth rate is proportional a dimensionless factor $1 - \dfrac{P(t)}{K}$, that is

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = r\left(1 - \frac{P(t)}{K}\right)$$

with r, K been nonnegative constants. Since $P \geq 0$, r is the maximal per-capita rate, referred to as the **intrinsic rate**. For $P < K$, the per-capita rate P'/P is positive, and the population increases. For $P > K$, $P'/P < 0$, and the population decreases. For this reason, K is called the **carrying capacity**.



The model is usually presented in the following standard equational form

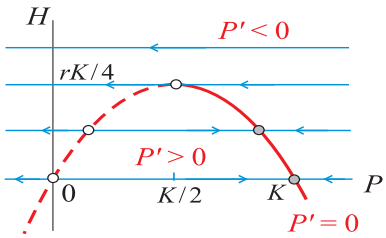
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

By a phase line analysis, we find that $P = 0, K$ are the only equilibrium points, for which $P = 0$ is a source and $P = K$ is a sink. Thus, for any nonvanishing initial population P_0, the solution through P_0 at $t = 0$ converges to K, i.e., the population eventually stabilizes at its carrying capacity K.

above the branch. The branch has a maximum point in H. It is found by elementary calculus to be

$$(\bar{P}, \bar{H}) = \left(\frac{K}{2}, \frac{rK}{4}\right).$$

The parameterized phase lines are as shown. Notice the qualitative differences for harvesting rate H above and below $\bar{H}$. $\bar{H}$ is a bifurcation value and $(\bar{P}, \bar{H})$ is a bifurcation point by definition.



More precisely, the bifurcation point $(\bar{P}, \bar{H})$ of this example is a prototypical case of a saddle-node bifurcation. It is characterized by the following.

Definition 2.2 *An equilibrium point* $(\bar{x}, \bar{\lambda})$ *of*

$$x' = f(x, \lambda)$$

is a ***saddle-node bifurcation point*** *if it satisfies the following conditions:*

- *For λ from one side of $\bar{\lambda}$, the equation has two equilibrium points, one stable and one unstable. Both equilibrium points emerge at $\bar{x}$ as λ converges to $\bar{\lambda}$*
- *For λ from the other side of $\bar{\lambda}$, the equation does not have an equilibrium point near $\bar{x}$. That is as λ crosses $\bar{\lambda}$ into this region, the two equilibrium points emerge at $\bar{x}$ and then disappear.*

The mathematical results of the example above have the following biological interpretations.

- For $H < \bar{H}$, the right equilibrium point is the continuation of the carrying capacity, and it is stable. The equilibrium state decreases when the harvest rate H increases. Thus this branch right of the bifurcation state $\bar{P}$ is called the **harvest mediated carrying capacity**.
- For $H < \bar{H}$, the left equilibrium state is the continuation of the non-existence state $P = 0$, and it is unstable. In contrast, it increases when H increases. If the population starts below the left equilibrium, it will reach the extinction state $P = 0$ in a finite time. If it starts above the equilibrium, it converges to and stabilizes at the harvest induced carrying capacity. For this reason, the equilibrium branch left of $\bar{P}$ is called the **harvest mediated survival threshold**.
- Because the threshold branch increases and the capacity branch decreases as H increases, the two branches head to each other. In this case they co... at the bifurcation point $\bar{P}$, which is also referred to as a **crash fold** point. For the harvest rate greater than the critical value $\bar{H}$, the population crashes down and eventually dies out, regardless its initial population.

Transcritical Bifurcation.

Example 2.2.3 Consider the logistic model with a constant *per-capita* harvest rate,

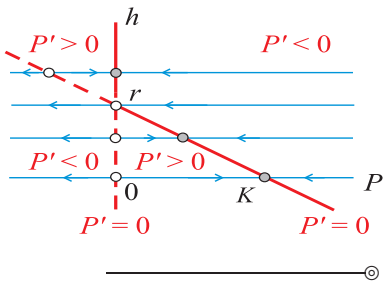
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - hP \tag{2.4}$$

where as before r is the maximum per-capita growth rate, K is the carrying capacity, and h is the constant per-capita harvest rate. Sketch the phase lines using h as the parameter.
Solution: The equilibrium branch is defined by $rP(1 - \frac{P}{K}) - hP = 0$. Unlike the constant harvest case, it is solved into two branches:

$$P = 0 \quad \text{and} \quad r\left(1 - \frac{P}{K}\right) - h = 0.$$

The latter defines a line, having $h = r$ as the h-intercept and $P = K$ as the P-intercept.

By inspection, we see that $(\bar{P}, \bar{h}) = (0, r)$ is a bifurcation point. Specifically, the threshold branch $P = 0$ changes its stability from unstable to stable as h increases through r. And the capacity branch through $(K, 0)$ changes its stability from stable to unstable.



The bifurcation point $(\bar{P}, \bar{h}) = (0, r)$ of this example is a prototypical case of a transcritical bifurcation. It is characterized by the following.

Definition 2.3 *An equilibrium point* $(\bar{x}, \bar{\lambda})$ *of*

$$x' = f(x, \lambda)$$

is a ***transcritical bifurcation point*** *if it satisfies the following conditions:*

- *There are two branches of equilibrium points intersect at* $(\bar{x}, \bar{\lambda})$ *at a nonvanishing angle.*
- *The equilibrium point on any branch of the two exchanges stability as the parameter* λ *passes through* $\bar{\lambda}$.

In the context of population dynamics, the transcritical bifurcation point $(0, r)$ of the example above is called the **capacity transcritical point**. In some cases where the transcritical bifurcation takes place on the survival threshold branch, the point is called the **threshold transcritical point** instead.

Exercises 2.2

1. Sketch the phase lines of the equations; determine the stabilities of the equilibrium points; and sketch the solution portraits in the tx-plane as well.

   (i) $x' = -x^3 + 2x^2 - x$

   (ii) $x' = -x^3 + 2x^2 - x + 0.5$

   (iii) $x' = 2\sin(\pi x)$

2. Find all equilibrium points, and determine their stabilities by the derivative test. Verify your answer schematically by sketch the phase lines.

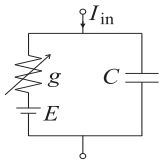
   (i) $x' = x^3 - x^2 - 2x$

   (ii) $x' = x - x^2 - \dfrac{0.3x}{0.25 + x}$ for $x \geq 0$.

3. Sketch the bifurcation diagram of this one-parameter family of differential equations

$$x' = f_a(x) = ax - x^3.$$

The type of bifurcation points this example represent is called **Pitch-fork Bifurcation**.



Problem 4

4. Consider the RC-circuit shown with an 'N' nonlinear IV characteristics $I = F(V) = V^3 - 2V^2 + V$. It is model by this one-parameter family of equations

$$C\frac{dV_C}{dt} = -F(E + V_C) - I_{in}$$

with the forcing current I_{in} being the parameter. Sketch the bifurcation diagram of the equations, and identify the type of bifurcation.

5. Consider the predator-prey model

$$x' = x(1 - x) - \frac{x}{\beta + x}y$$

for which the predator population density y is considered as a parameter. For a fixed vale

$$0 < \beta < 1$$

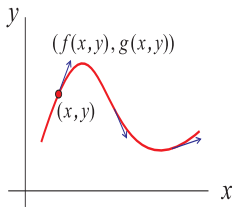
sketch the bifurcation diagram in the first quadrant $x \geq 0, y \geq 0$ with y being a parameter.

2.3 PHASE PLANE METHOD

We now extend the phase line method to autonomous systems of two equations

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{2.5}$$

where f, g are nice functions which guarantee the uniqueness of solutions to initial value problems.



In the (x,y) phase space, a solution gives rise to an orbit $\{(x(t), y(t))\}$ as a curve parameterized by the time variable t. At each point of the orbit, it moves in the direction and speed given by the velocity vector $(x'(t), y'(t))$, which is prescribed by the vector field $(f(x(t),y(t)), g(x(t),y(t)))$. That is, an orbit is not just any curve in the phase space. It must follow the vector field $(f(x,y), g(x,y))$ at every point (x,y) in the space. See illustration.

The description above gives rise to the following approximation of orbits. We generate a mesh grid in x, y with a user defined mesh size. At each grid point (x_i, y_j), we plot the vector field $(f(x_i,y_j), g(x_i,y_j))$. Then an orbit starting at any point inside the mesh can be sketched by following the vector field as closely as possible.

Though the method above is suitable for computers, it does not make a good use of human intuitions. This is where the phase plane method enters. It gives an insightful and systematic way to organize the vector field. In some cases it gives a rather comprehensive, qualitative understanding on the orbital structure of the equations. In the cases it fails to be complete, it nevertheless gives a good, first order approximation to the structure.

The Method

Like the phase line method, the phase plane method again makes use of equilibrium points to organize the vector field. Particularly indispensable are the curves defined by $f(x,y) = 0$ and $g(x,y) = 0$. If one thinks y as a parameter, the former $f(x,y) = 0$ would define the equilibrium branch for the x-equation. Symmetrically, if one thinks x as a parameter, the latter $g(x,y) = 0$ would define the equilibrium branch for the y-equation. For these reasons, we call the curve $f(x,y) = 0$ the **x-nullcline** and $g(x,y) = 0$ the **y-nullcline**.

Here is a brief description for the procedures.

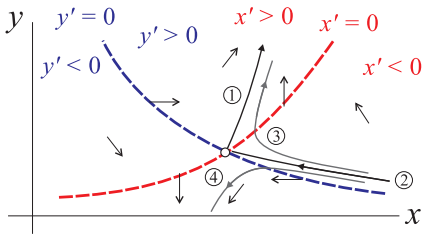
1. In the (x,y) phase space, sketch the x-nullcline $f(x,y) = 0$. It divides the space into regions of $f > 0$ and $f < 0$. Use the **one-point testing** technique to label these regions.
2. Do the same for the y-equation: sketch the y-nullcline $g(x,y) = 0$ and identify the regions with $g > 0$ and $g < 0$.
3. On each segment of the x-nullcline partitioned out by the equilibrium points, place a vertical vector filed $(x', y') = (0, g)$. The vector points

up if the segment lies in a $y' = g > 0$ region, and points down if otherwise.

4. Do the same for the y-nullcline, in reverse. That is one each segment of the y-nullcline, place a horizontal vector filed $(x', y') = (f, 0)$, pointing right if it lies in a $x' = f > 0$ region or left if it lies in a $f < 0$ region.
5. In each region bounded by x-nullcline and y-nullcline, the vector field (f, g) never vanish in either component. Place a vector inside the region according to the following convention

sign(x', y'):	$[+,+]$	$[+,-]$	$[-,+]$	$[-,-]$
vector field (x', y'):	↗	↘	↖	↙
description:	right-up	right-down	left-up	left-down

6. Sketch any special as well a few typical orbits following the directions of the vector field in the regions bounded by the nullclines. In the illustration below, for example, orbit 1 coming out the equilibrium point in the $[+,+]$ region is such a special orbit, called **separatrix**. It moves right-up in the region. Orbit 2 is another. Orbits 3 and 4 are typical ones. Notice that orbit 3 must be drawn to move left-up in the $[-,+]$ region, cross the x-nullcline vertically, turn around in the x-component and move right-up in the $[+,+]$ region. In contrast, when orbit 4 crosses the y-nullcline, it must do so horizontally.



As one can see in the illustration above, one can fill other special and typical orbits in the $[+,-]$ and $[-,-]$ regions to complete the steps of the method. Also, in this hypothetical case, the orbital structure near the equilibrium point shown is completely understood — it has the structure of a saddle point.

A word of caution for beginners. Equilibrium points lie always on nullclines, but only in rare situations a nullcline becomes a solution to the differential equations.

Phase Portrait of 2-D Linear Systems.

To illustrate the method, we first use it on a linear system to show how the phase portraits of such systems can be completely understood especially

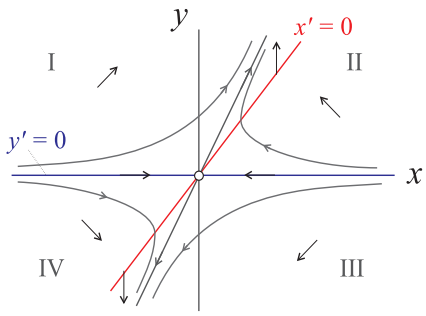
when combining with the information of their eigensolutions.

Example 2.3.1 Recall the linear system of equations (2.1.3) from Sec. 2.1 together with their linearly independent eigensolutions:

$$x' = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} x, \ \mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}, \ \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}.$$

To sketch a phase portrait by the phase plane method, we first sketch the x-nullcline: $x' = -2x + 3y = 0$, and the y-nullcline: $y' = 4y = 0$ as show in the figure. These lines divide the plane into 4 regions labelled as I, II, III, IV for which the vector field (x', y') is annotated below both in the table and the graph.

	I	II	III	IV
x'	+	−	−	+
y'	+	+	−	−
vector field	↗	↖	↙	↘



The vector field is also annotated on the nullclines as shown. To determine the signs of x', y' in the region I, for example, we only need to pick a simple point from the region, say $(0, 1)$ from the y-axis, and we find $x' = f(0, 1) = 3 > 0$ and $y' = g(0, 1) = 4 > 0$, and hence the same signs for x' and y' respectively throughout the region. In region II, we may pick a point $(N, 1)$ with a large $N > 0$, which in turn gives $x' < 0$ and $y' > 0$. The direction of the vector field on the nullclines can also be determined similarly. Notice that the y-nullcline is special — it happens to be the contracting eigensolutions. The expanding eigensolutions fall inside regions I and III, not crossing any nullcline lines. In contrast all other solutions either lie on or cross one of the nullclines. Notice also, even without the exact eigensolutions, their qualitative properties can be deduced from the phase portrait alone and so can be the saddle property of the trivial equilibrium solution. ◎

Dynamics of Competition.

We now apply the method to nonlinear systems of equations. Consider a population model of two competing species x, y, each separately is governed by a logistic growth, each's per-capita growth rate is diminished at a rate proportional to the other's presence. The model is as follows

$$\begin{cases} x' = r_x x \left(1 - \dfrac{x}{K_x}\right) - c_y xy = x \left[r_x \left(1 - \dfrac{x}{K_x}\right) - c_y y\right] := x f(x, y) \\ y' = r_y y \left(1 - \dfrac{y}{K_y}\right) - c_x yx = y \left[r_y \left(1 - \dfrac{y}{K_y}\right) - c_x x\right] := y g(x, y). \end{cases} \tag{2.6}$$

Here r_i, K_i are the intrinsic rates and carrying capacities, and $c_j > 0$ are the per-capita competition coefficient of species j against species i.

Species x is said to be **competitive** if its per-capita growth rate is positive at the y-capacity point $(0, K_y)$, that is

$$\frac{1}{x}\frac{dx}{dt}\Big|_{(0,K_y)} = f(0, K_y) = r_x - c_y K_y > 0 \iff \boxed{K_y < \frac{r_x}{c_y}.}$$

Notice that according to this characterization, species x is competitive if species y's per-capita adversary impact coefficient c_y is relatively small. Similarly, y is competitive if it can grow per-capita at the x-capacity point $(K_x, 0)$ so that

$$\boxed{K_x < \frac{r_y}{c_x}.}$$

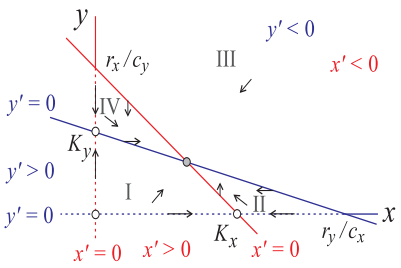
Example 2.3.2 (**Competitive Coexistence.**) Use the phase plane method to analysis the dynamics of the competition model Eq.(2.6) when both species are competitive, i.e.,

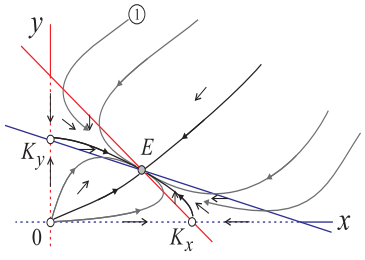
$$K_x < r_y/c_x \quad \text{and} \quad K_y < r_x/c_y.$$

Solution: We begin by sketch the x-nullcline: $xf(x, y) = x[r_x(1 - x/K_x) - c_y y] = 0$ which are separated into two branches

$$x = 0 \quad \text{and} \quad r_x(1 - x/K_x) - c_y y = 0.$$

The latter is the y-competition mediated x-carrying capacity: along which x decreases as y increases, having the x-intercept at the x-carrying capacity $(K_x, 0)$, and the y-intercept at the x-capacity transcritical point $(0, r_x/c_y)$. See the illustration.





Similarly, the y-nullcline $yg(x, y) = y[r_y(1 - y/K_y) - c_x x] = 0$ has two branches

$$y = 0 \quad \text{and} \quad r_y(1 - y/K_y) - c_x x = 0,$$

with the latter being the x-competition mediated y-carrying capacity: along which y decreases as x increases, having the y-intercept at the y-carrying capacity $(0, K_y)$, and the x-intercept at the y-capacity transcritical point

$(r_y/c_c, 0)$. Because of the dual competitive assumption, the x-capacity transcritical r_x/c_y is greater than the y-capacity K_y on the y-axis and the y-capacity transcritical r_y/c_x is greater than the x-capacity K_x on the x-axis as shown.

Next, we label the subregions in the first quadrant that are bounded by the nullclines as I, II, III, IV. The signs of f, g in these regions are as shown below. The signs are determined by one-point test technique. For example, in region

	I	II	III	IV
x'	$+$	$-$	$-$	$+$
y'	$+$	$+$	$-$	$-$
vector field	$\nearrow$	$\nwarrow$	$\swarrow$	$\searrow$

I, the sign of x' can be determined by evaluating $x' = xf(x, y)$ at any point in the region. Taking $(K_x/2, 0)$ for convenience, we find $x'|_{(K_x/2,0)} = (K_x/2)f(K_x/2, 0) > 0$ and hence a '+' sign in the table.

Having determined the directions of the vector field (x', y'), a representative vector field is placed in each of the region.

One the nullclines, the vector field is either horizontal or vertical. For example, on the part of the y-nullcline which forms the common boundary of region I and region IV, x' is positive. Hence, it is given a right arrow.

Last, we are ready to sketch special and typical orbits. First, equilibrium orbits are simply the intersections of the x-nullcline and the y-nullcline. There are 4 of them, the trivial mutual extinct state $(0, 0)$, the competition-free individual capacity states $(K_x, 0), (0, K_y)$, referred to as the x-equilibrium and the y-equilibrium respectively, and the coexistent state referred to as the xy-equilibrium. The xy-equilibrium point can be solved explicitly from $f(x, y) = 0, g(x, y) = 0$ if needed.

There are at least four special, non-equilibrium orbits, one in each region I, II, III, IV, connecting equilibrium orbits without crossing the nullclines.

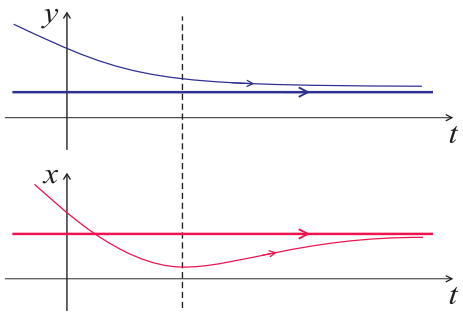
Last, sketch a few typical solutions to complete the phase portrait. For example, starting at any point in region III, below the special orbit converging to E, the orbit must develop left and down, cross the y-capacity nullcline horizontally, turn upward but still move left, remain in region II thereafter and tend to E. E is asymptotically stable.

We can now conclude that starting at nonvanishing populations for both species $x_0 > 0, y_0 > 0$, the orbit converges to the equilibrium point E. The biological significance of this result is that mutually competitive species can coexist at an equilibrium state in a long run.

◎

We have used two figures in the example above to highlight the steps. You can certainly plot the orbital structure from the right figure onto the left one if it does not become too cluttered.

Again like the case of phase lines, the phase portrait of a system of equations captures all essential geometric properties of the solutions to the system. For example, one can qualitatively construct the time series of a solution from its orbit in the phase plane. The times series on the right are reconstructed for Orbit 1 from the phase portrait of the example above. We note that the vertical dash line indicates the time at which the orbit crosses the x-nullcline before its converging to the coexisting equilibrium point.



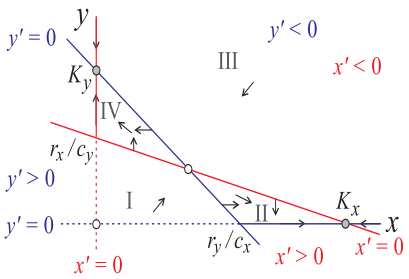
Example 2.3.3 (**Competitive Tug of War.**) Use the phase plane method to analysis the dynamics of the competition model Eq.(2.6) when both species are not competitive, i.e.,

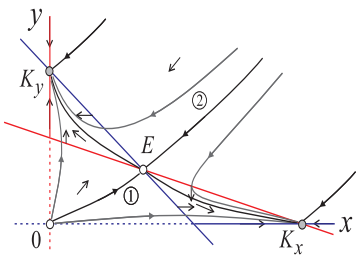
$$K_x > r_y/c_x \quad \text{and} \quad K_y > r_x/c_y.$$

Solution: We follow the same steps and analysis as in the previous example. The critical difference in result is that the the x-capacity branch and the y-capacity branch switch their relative position with each other, in particular the competition free capacities K_i switch position with the capacity transcritical point r_j/c_i on the axis i. As a result, the vector field are drastically different in subregion II and IV as shown in the table below. By making

	I	II	III	IV
x'	+	+	−	−
y'	+	−	−	+
vector field	↗	↘	↙	↖

appropriate changes to the previous plot, a phase portrait is shown below. Again one can use just one plot by including the orbits in the left figure if it is not too cluttered.





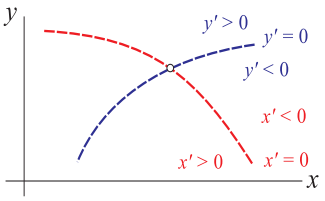
Orbit 1 and orbit 2 are **separatrixes**. They divide the first quadrant into two parts. The region above the separatrixes is the non-competitive region for x, in which all orbits converge to the y-capacity state $(0, K_y)$ with x driven to extinction. The region below the separatrixes is the non-competitive region for y, in which all orbits converge to the x-capacity state $(K_x, 0)$ with y driven to extinction. In other words, the outcome depends on the initial state of the population. Initial states on the separatrixes or the equilibrium state E cannot persist. Any small perturbation will throw the states into one of the non-competitive regions, driving off one competitor. There is no coexisting state.

The remaining case with one species competitive and the other not is left as an exercise, which leads to the phenomenon of **competitive exclusion**.

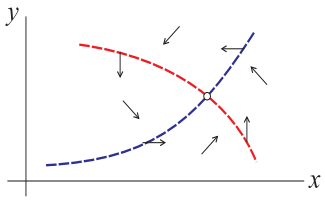
Exercises 2.3

1. The nullclines and some information about the vector field for a system are given in each of the diagram. Use the phase plane method to sketch a phase portrait of each system.

(a)



(b)



(Remark: Problem (b) would expose a weakness of the method.)

2. Consider again Problem 2 of Exercise 2.1. Refine the phase portraits of the equations by incorporating the phase plane method.

(i) $x' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} x$

(ii) $x' = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x$

(iii) $x' = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} x$

3. Consider the population model of two competing species Eq.(2.6). Use the phase plane method to sketch a phase portraits for the case in which one species is competitive and the other is not. The dynamics outcome of such a case is referred to as **competitive exclusion**.

4. Consider the population model of two cooperative species x, y:

$$\begin{cases} x' = r_x x\left(1 - \dfrac{x}{K_x}\right) + b_y xy \\ y' = r_y y\left(1 - \dfrac{y}{K_y}\right) + b_x yx \end{cases}$$

where all parameters r_i, b_j are positive.

(a) Show that it has an equilibrium point $E = (E_x, E_y)$ with positive components $E_x > 0, E_y > 0$ under the following condition

$$b_x b_y < \frac{r_x r_y}{K_x K_y}$$

(b) Use the phase plane method to sketch a phase portrait of this case.

2.4 PHASE PLANE WITH FAST AND SLOW TIME SCALES

The dynamics of a system of equations is determined by its vector field. The phase plane method takes into account the direction of the vector field but ignores so far its magnitude. This works well for some cases such the ones considered in the previous section. In this section, however, we consider systems for which the magnitude of their vector fields is too obvious to ignore. Such are systems having different time scales.

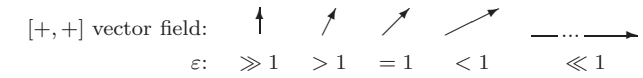
Consider systems of the following form

$$\begin{cases} \varepsilon x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{2.7}$$

where $0 < \varepsilon \ll 1$ is 'very' small parameter as signified by the notation '$\ll 1$'. Looking at it from the point view of vector field, it is more convenient to write it as

$$\begin{cases} x' = \dfrac{f(x,y)}{\varepsilon} \\ y' = g(x,y) \end{cases}$$

We see immediately that the magnitude of the vector field changes dramatically with ε: if $f(x,y) \neq 0$ at a point (x,y), the x-component of the vector field is extremely large for very small $\varepsilon \ll 1$, or extremely small for very large $\varepsilon \gg 1$. The table below shows the effect of ε on the vector field (x', y') in the case when both f and g are positive.



At one extreme when $\varepsilon \sim \infty$, the vector field is nearly vertical $(0, g)$. At the other extreme when $\varepsilon \sim 0$, the vector field is nearly horizontal $(f/\varepsilon, g)$ with a near infinity magnitude in the horizontal direction.

Terminology: Systems of the form (2.7) with $0 < \varepsilon \ll 1$ are referred to as **singularly perturbed** systems and the parameter ε a **singular parameter**. Since the rate of change for the x variable is large with f/ε relative to the rate of change for the y variable, variable x is called the **fast variable** and variable y is called the **slow variable**.

The equation form (2.7) can be cast in an alternative form by the following change of time variable

$$\tau = \frac{t}{\varepsilon} \quad \text{so that} \quad d\tau = \frac{dt}{\varepsilon}$$

By the chain rule,

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \frac{dx}{dt}\varepsilon, \quad \text{and similarly,} \quad \frac{dy}{dt} = \frac{1}{\varepsilon}\frac{dy}{d\tau}.$$

Use these relations to rewrite Eq.(2.7) as

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = \varepsilon g(x,y) \end{cases} \tag{2.8}$$

where $\dot{z} = \dfrac{dz}{d\tau}$.

Because of the conversion relation $\tau = t/\varepsilon$, t is called the **fast time** and τ is called the **slow time**. For example, if the unit of t is in year, and $\varepsilon = 10^{-3}$, then 1 year in the t time scale equals 1000 years in the τ time scale. Correspondingly, the original form Eq.(2.7) is the singularly perturbed system in the **fast time scale** and the equivalent form Eq.(2.8) is the system in the **slow time scale**.

Time Scale Method.

The theory of singular perturbation is about how dynamics of singularly perturbed systems change with their singular parameters. Here we introduce the most elementary geometric method of the theory. It incorporates the the diverging magnitude of a singularly perturbed vector field into the phase plane method. Thus, we may call it **phase plane method with time scales** or simply the **time scale method**. Essential to the method are two types of orbits at the singular limit $\varepsilon = 0$. We begin with the simpler kind.

Fast Orbits.

This orbit type is associated with the *slow* time scale system (2.8) at the singular limit $\varepsilon = 0$:

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = 0. \end{cases} \tag{2.9}$$

It is called the **fast subsystem** of the perturbed systems (2.7,2.8). This may first appear paradoxical and confusing. However, upon a further reflection, it makes a perfect sense — a fast moving object is best analyzed in a slow motion time scale.

The key realization that drives the analysis for this subsystem is that because x moves so fast, variable y appears frozen in time. The equation $\dot{y} = 0$ says it all: y stays at its initial state wherever it starts, and therefore it can be thought as a parameter one value a time. Hence, the orbital structure of the subsystem is completely determined by the phase line method with parameter from Sec.2.2. For illustration purposes, let the x-equilibrium branch (more accurately the x-nullcline for the full system), $f(x, y) = 0$ be as in Fig.2.5(a) together with phase lines parameterized by the slow variable y. The phase line x-orbits are called the **fast orbits**.

Algebraic-to-Differential Reduction Method for Slow Solutions. One way to find a solution $(x(t), y(t))$ with an initial point $(x(0), y(0)) = (x_0, y_0)$ to the slow subsystem (2.10) is to follow the steps below.

1. Verify that point (x_0, y_0) is on the x-nullcline, satisfying $f(x_0, y_0) = 0$.
2. Check if variable x can be uniquely solved as a function of variable y, i.e. $x = x(y)$, from the algebraic equation $f(x, y) = 0$ in a small open interval containing y_0 satisfying $x(y_0) = x_0$.
3. If both conditions above are satisfied, then solve the reduced initial value problem
$$\begin{cases} y' = g(x(y), y) \\ y(0) = y_0 \end{cases}$$
4. If $y(t)$ is a solution, then a solution to the slow subsystem is found by backward substitution as $(x(y(t)), y(t))$.

Whether or not this analytical method works critically depends on the second condition above. By definition, a point (x_*, y_*) is called a **turning point** if it satisfies $f(x_*, y_*) = 0$ but x cannot be uniquely solved from $f(x, y) = 0$ as a function of y. Hence, the solution procedure above fails at a turning point. As a result the slow subsystem may or may not have a solution, or if it does, the solution may not be a unique solution.

Turning points are not new. We have encountered them in a different disguise. In fact, we have the following duality.

Turning Points are Bifurcation Points:

> A point (x_*, y_*) is a turning point of the slow subsystem (2.10) if and only if it is a bifurcation point of the fast subsystem (2.9).

Slow Orbits.

This orbit type is associated with the *fast* time scale system (2.7) at the singular limit $\varepsilon = 0$:

$$\begin{cases} 0 = f(x, y) \\ y' = g(x, y) \end{cases} \tag{2.10}$$

It is called the **slow subsystem** of the perturbed systems (2.7,2.8). Again, similarly to the reason of analyzing the fast dynamics at the slow time scale, it is best to analyze slowly moving objects at a fast time scale.

The key realization that drives the analysis for this subsystem is that its orbits must lie on the x-nullcline, $f(x, y) = 0$, and the dynamics is driven primarily by the y-equation. Since the equation is only one-dimensional on the x-nullcline curves, the phase line method can be adapted onto the restricted curves. As a result the dynamical structure is rather simple.

Slow Solution Structures. The orbits of the slow subsystem can only behave in one of the following three ways:

- Stay at an equilibrium state.
- Monotone increase or decrease toward an equilibrium state.

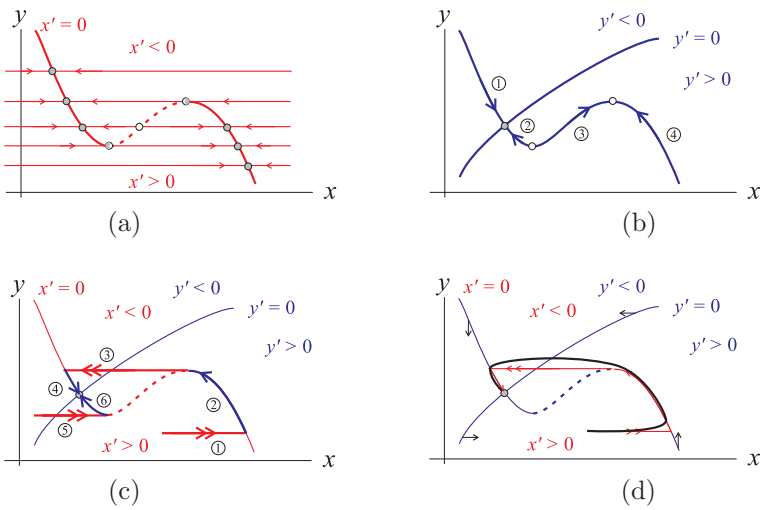


Figure 2.5 Time Scale Method

- Monotone increase or decrease toward a turning point of the system that defines a boundary point of the x-nullcline branch to which the slow subsystem is restricted.

In other words, the structure is completely determined by equilibrium points and turning points. It is this geometric approach that we will use predominantly throughout.

The phase line method for the slow subsystem is illustrated as in Fig.2.5(b) which is the continuation of the hypothetical case of Fig.2.5(a). The directed curves on the x-nullcline are the orbits of the slow subsystem. They are called the **slow orbits**. This hypothetical illustration display all the slow orbit types listed above:

- The intersection of x-nullcline and y nullcline is automatically an equilibrium orbit of the slow subsystem.
- Orbits 1 and 2 converge to the equilibrium point.
- The turning points are saddle-node bifurcations at which the slow subsystem ceases to be well-defined. Orbits 3 and 4 head toward the right turning point, reaching it at a finite time because of the nonvanishing velocity of the y variable, and out of bound thereafter.

The reduced equations that define orbits 3 and 4 are completely different from each other. That is why the turning point must not be falsely perceived as an equilibrium point. For the same reason, the left turning point is not an equilibrium point. It deceptively looks like a source, which is a boundary point of two different branches of the slow subsystem.

Why Singular Orbits? The answer lies in the fact that orbits of the perturbed full system (2.7) with $0 < \varepsilon \ll 1$ converge to singular orbits as $\varepsilon \to 0$. More specifically, let

$$(x^\varepsilon(t), y^\varepsilon(t))$$

denote an orbit of Eq.(2.7) having the same initial point

$$(x^\varepsilon(0), y^\varepsilon(0)) = (x_0, y_0)$$

for all $0 < \varepsilon \ll 1$. Then the limit $\lim_{\varepsilon\to 0}(x^\varepsilon(t), y^\varepsilon(t))$ is expected to exist, and the limit is a singular orbit through the same initial point. In other words, singular orbits are the 0th order approximation of the dynamical structure of the system when ε is small.

Singular Orbits.

Incorporating both slow and fast orbits in one figure, we obtain the phase portrait of the systems (2.7,2.8) at the singular limit $\varepsilon = 0$. For the illustrative example, it is shown in Fig.2.5(c). By definition, the concatenation of fast and slow orbits with a congruent orientation is called a **singular orbit**. Shown as examples, the ordered concatenation of singular orbits 1, 2, 3, 4 is one singular orbit. So is the combination with 2, 3, 4. Orbits 5 and 6 form another. Equilibrium points of the full system is a trivial kind of singular orbits.

Fig.2.5(d) illustrates how a perturbed orbit (with $0 < \varepsilon \ll 1$) of the singular orbit $\{1, 2, 3, 4\}$ (with $\varepsilon = 0$ from (c)) may look like, starting at the same initial point. Notice that, the orbit's profile must also obey the vector field behavior deduced from the phase plane method. For example, it still moves right-up in the region where $x' > 0, y' > 0$, but more flatly so. It crosses the x-nullcline vertically, and the y-nullcline horizontally, respectively.

The time scale method completes the analysis of a singularly perturbed system with the phase portrait of special and typical singular orbits. The singular orbit structure is considered as the 0th order approximation of the perturbed structure for small $0 < \varepsilon \ll 1$ (see the insert discussion on "Why Singular Orbits").

Example 2.4.1 Consider the same competitive model of two species as (2.6) except for the dimensional notation for the population densities X, Y

$$\begin{cases} X' = r_x X\left(1 - \dfrac{X}{K_x}\right) - c_y XY \\ Y' = r_y Y\left(1 - \dfrac{Y}{K_y}\right) - c_x XY. \end{cases}$$

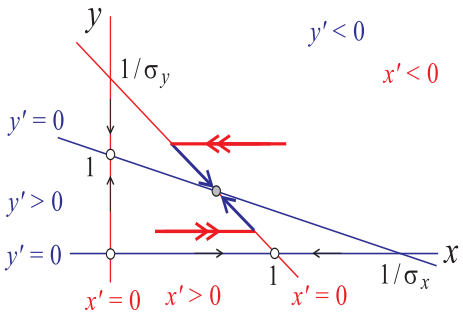
It is left as an exercise to show that with the following change of variables and parameters

$$x = \frac{X}{K_x}, \qquad y = \frac{Y}{K_x}, \qquad s = r_y t$$

$$\sigma_y = \frac{c_y K_y}{r_x}, \qquad \sigma_x = \frac{c_x K_x}{r_y}, \qquad \varepsilon = \frac{r_y}{r_x},$$

the system is transformed into the following dimensionless form

$$\begin{cases} \varepsilon \dfrac{dx}{ds} = x\,(1-x) - \sigma_y xy := x f(x,y) \\ \dfrac{dy}{ds} = y\,(1-y) - \sigma_x xy := y g(x,y). \end{cases}$$

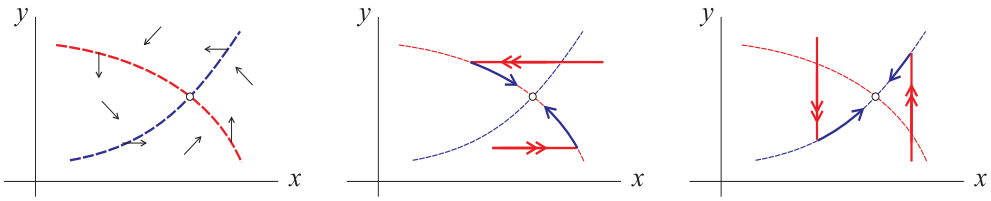


The dimensionless system becomes a singularly perturbed system if $0 < \varepsilon \ll 1$, i.e., species X is much more prolific than species Y is. Under the competitive coexistence condition that

$$\sigma_x,\ \sigma_y < 1$$

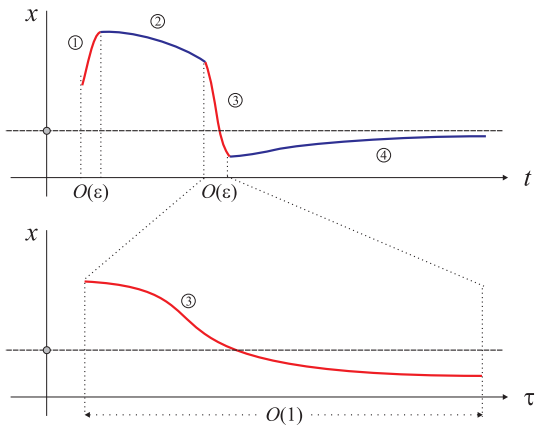
the phase portrait of singular orbits is sketched in the figure. Notice that all singular orbits not originated from either axis converge to the xy-equilibrium point.

Comparison of Phase Plane and Time Scale Methods. The time scale method is based on the phase plane method and compensates the latter's shortcoming of ignoring vector field's magnitude. The illustration below gives an example of this point. Figure (a) is a phase plane illustration in which only the direction not the magnitude of a vector field is depicted. Following the vector field, you draw an orbit circling around the equilibrium point at the best. One cannot conclude if such an orbit converges to the equilibrium point, or diverges from it, or stays on a cycle. However, if we know the system is singularly perturbed like equations (2.7), then the singular orbit structure at $\varepsilon = 0$ looks like (b), for which all singular orbits converge to the equilibrium point. Similarly, if we know the parameter ε is not a zero singularity rather an infinity singularity that $\varepsilon \gg 1$ so that x is slow and y is fast, then the singular orbit structure at $\varepsilon = \infty$ looks like (c). Again, all singular orbits converge to the equilibrium point.



(a) (b) (c)

Comparison of Time Scales. The illustration on the right gives a comparison between the fast time scale t and the slow time scale τ as to how they may shape the time series profiles of singular and perturbed orbits. The orbit in consideration is the perturbed orbit from Fig.2.5(d) with its four fast and slow phases (1, 2, 3, 4) corresponding to those of Fig.2.5(c). In the fast time scale plot against the x variable, the sharp rise (1) and fall (3) become instantaneous jumps when $\varepsilon = 0$ which are referred to as **phase transitions**. The relaxed transitions with $0 < \varepsilon \ll 1$ take place in a t-time interval of order ε, $O(\varepsilon)$, which shrink to an instantaneous moment as $\varepsilon \to 0$.



In the slow time scale τ, however, these short transitions are slowed down and magnified. So much so that at $\varepsilon = 0$, the magnified τ-interval becomes infinity, and the fast orbit is stretched indefinitely to the right. It can only be done one fast orbit a time.

Dynamical System Evolution to Higher Dimensions. The most important and useful feature of the time scale method is the dimensional reduction property: both fast and slow subsystems are at least one dimension less than the full system. Hence, the method breaks the system down to lower dimensional slow and fast subsystems, unlock their full structures, only to build them up to construct a 0th order approximation of the full system, which in most cases give a qualitatively accurate description of the system for small perturbations from its singularity.

The time scale method is a reductionistic approach at its best. It allows us to understand higher dimensional structures from their lower dimensional components.

Exercises 2.4

1. Show that the changes of variables and parameters from Example (2.4.1) transform the dimensional system to the dimensionless system as claimed. (*Hint:* Use the chain rule of differentiation. For example,
$$\frac{dX}{dt} = \frac{dX}{ds}\frac{ds}{dt} = K_x r_y \frac{dx}{ds}.$$
Substitute this into the X-equation and simplify to get the dimensionless counterpart. Do the same for the Y-equation.)

2. Consider the same dimensionless competition model from Example (2.4.1). Use the time scale method to sketch a phase portrait of singular orbits for the following cases

   (a) Competitive Tug of War: $\sigma_x,\ \sigma_y > 1$.

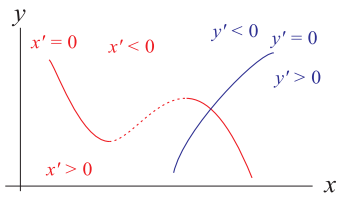
   (b) Competitive Exclusion: $\sigma_x < 1 < \sigma_y$.

3. The nullclines and some information about the vector field for a system are given in each of the diagram.

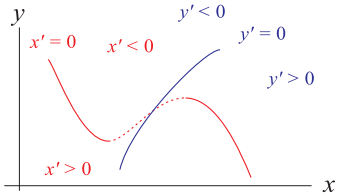
   (a) Use the time scale method to sketch typical singular orbits, assuming x is the fast variable.

   (b) Use the time scale method to sketch typical singular orbits, assuming y is the fast variable.

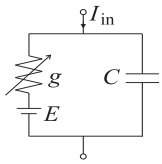
(i)



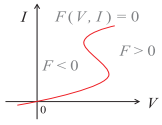
(ii)



4. Consider the nonlinear RC circuit with an S-shaped IV-characteristic $F(V_g, I_g) = 0$ for the resistor g. The following singularly perturbed system models the circuit dynamics.



$$\begin{cases} C\dfrac{dV_C}{dt} = -I_g - I_{in} \\ \varepsilon\dfrac{dI_g}{dt} = F(V_C + E). \end{cases}$$



Use the time scale method to sketch the singular phase portrait of the system under the following condition:

(a) The V_C-nullcline lies below the lower knee point of the S-characteristics.

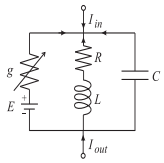
(b) The V_C-nullcline lies between the lower and upper knee points of the S-characteristics.

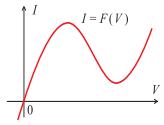
(c) The V_C-nullcline lies above the upper knee points of the S-characteristics.

2.5 RELAXATION OSCILLATIONS

Perhaps the most useful advantage of the time scale method over the regular phase plane method is that it can be used to capture large limit cycles for systems of 2-dimension or higher. A **limit cycle** of a system is a periodic solution $x(t)$ satisfying $x(t+T) = x(t)$ for all t and for a fixed $T > 0$. The smallest of such positive T is called the **period** of the cycle. We consider two types of limit cycles of singularly perturbed systems. For the first type, the singular limit cycle contains saddle-node turning points only. For the second type, the singular limit cycle contains at least one transcritical turning point. We present them by examples. We begin with the saddle-node turning point case because it is simpler.

Singular Cycle Through Saddle-Node Turning Points.





We use the FitzHugh-Nagumo circuit as a prototypical example. The circuit and the nonlinear IV-characteristics are shown in the margin. The system of equations that models the circuit dynamics is given as follows.

$$\begin{cases} C\dfrac{dV_C}{dt} = -F(E+V_C) - I_L - I_{in} \\ L\dfrac{dI_L}{dt} = V_C - RI_L. \end{cases} \tag{2.11}$$

With the following change of the time variable, and introduction of a new parameter

$$t := \frac{t}{L}, \qquad \varepsilon = \frac{C}{L},$$

the circuit equations are transformed into

$$\begin{cases} \varepsilon\dfrac{dV_C}{dt} = -F(E+V_C) - I_L - I_{in} \\ \dfrac{dI_L}{dt} = V_C - RI_L. \end{cases} \tag{2.12}$$

Here we have used the same notation t for both the original time t and the new time t/L for conservation of notation.

The new parameter ε captures the energy storage capability of both the capacitor and the inductor. More specifically, from the capacitor relation $V = Q/C$ we see that the smaller C is, the greater energy the capacitor can store for the same amount of charge Q. Similarly, from the inductor relation $LI' = V$ we see that the larger L is, the greater potential energy the inductor can store for the same amount of change in the current. Hence, the magnitude of the ratio $\varepsilon = C/L$ conveys an unambiguous interpretation: the smaller ε is, the greater energy the two devices as a whole can store.

Let us now use the time scale method to analyze the circuit dynamics treating ε as a singular parameter. Recall, the phase plane method still

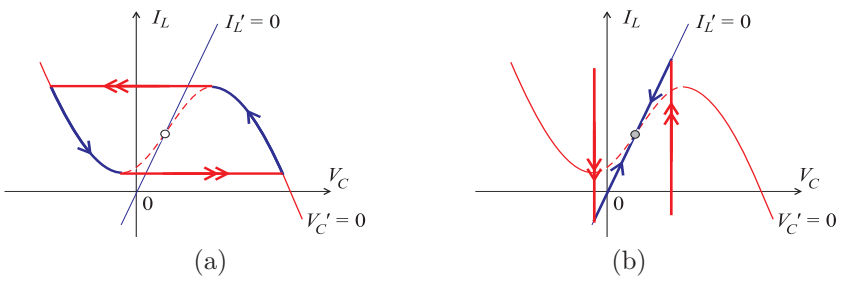


Figure 2.6

applies, but with the added information on fast and slow time scale dynamics when ε is either very small or very large.

To begin, I_L-nullcline is a line $V_C = RI_L$, through the origin with a positive slope R. The V_C-nullcline has the shape of an upside-down letter N, which is transformed from the N-shaped IV-characteristic $I = F(V)$ for the nonlinear resistor. To be precise, the V_C-nullcline is, $I_L = -(F(V_C + E) + I_{in})$.

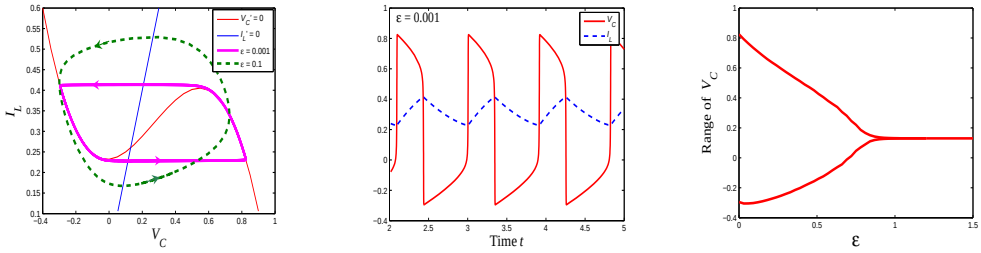
There are two broad cases regarding the parameter values of E and I_{in}. The circuit for which the parameter combination in E and I_{in} makes the I_L-nullcline, $V_C = RI_L$, to intersect *only* V_C-nullcline's middle branch between its two extreme points is said in an **excitable state**. The circuit for which the I_L-nullcline intersects either the left branch or/and the right branch of the V_C-nullcline is said in a **non-excitable state**.

We consider the excitable state as shown in Fig.2.6 and leave the non-excitable state to the Exercises. The singular phase portraits illustrate two cases: Fig.2.6(a): $0 < \varepsilon \ll 1$ for which V_C is the fast variable; Fig.2.6(b): $\varepsilon \gg 1$ for which I_L is the fast variable. In case (a) all orbits converge to a limit cycle, referred to as **relaxation oscillation**. In case (b) all orbits converge to an equilibrium point. That is, in terms of its energy storage capability, the circuit destabilizes into oscillation when its energy storage capability is high, and stabilizes at an equilibrium state when its energy storage capability is low.

Figure 2.7 shows some computational simulations of the circuit with

$$F(V) = aV(V^2 + bV + c),\ a = 2,\ b = -2,\ c = 1.1$$
$$R = 0.5,\ E = 0.4,\ I_{in} = -0.6$$

Fig.2.7(a) compares limit cycles for two small values of ε. It shows that the smaller ε is the tighter the cycle hugs round the singular cycle. Fig.2.7(b) shows the time series of both variables for one ε value. Fig.2.7(c) is a bifurcation diagram plot. It is generated as follows. We first discretize an ε interval into N many points, $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$. Then for each ε_k, an orbit is generated for a T period of time using the previous the end point of ε_{k-1}'s orbit as the initial point. We then keep only the last $T_0 < T$ period of the orbit, and find its maximal and minimal values in the voltage V_C. Fig.2.7(c) is the plot of the extremes against the parameter ε. To interpret, the nonzero variation



(a) (b) (c)

Figure 2.7

in the minimum and maximum implies that the asymptotic dynamics of the circuit is a limit cycle, and the lack of variation means an equilibrium point. The plot is generated by the Matlab m.file `FHNbifurcation.m`.

Singular Cycle Through Transcritical Turning Points.

Consider the following dimensionless predator-prey model which we will derive in detail in a later section

$$\begin{cases} \varepsilon x' = x(1-x) - \dfrac{x}{\beta + x} y \\ y' = y\left(\dfrac{x}{\beta + x} - \delta - \mu y \right). \end{cases} \tag{2.13}$$

Here x is the dimensionless population density of a prey and y is the dimensionless population density of a predator. Without the predator ($y = 0$), the prey grows according to the logistic model. The functional form

$$\frac{x}{\beta + x}$$

is the dimensionless kill rate per-capita of y. It is referred to as Holling's Type II functional. Parameter β is the **semi-saturation rate** in the sense that when $x = \beta$ the kill rate equals $1/2$, half of the dimensionless saturation kill rate $1 = \lim_{x\to\infty} \frac{x}{\beta+x}$.

The predator is a specialist in the sense that its existence depends exclusively on the prey x. The dimensionless growth rate is the same as the kill rate. Without it, it dies off at a per-capita rate $\delta + \mu y$, where μ is the **intraspecific** mortality rate due to competition amongst the predator.

The time scale parameter ε measures the ratio of reproduction rates of the predator and the prey. For small ε, the prey out produces the predator. For large ε, the predator out produces the prey. It is more intuitive to envision scenarios of the first case. The second case, however, is not that uncommon such as in cases of biological control, infectious diseases, and parasitic interaction.

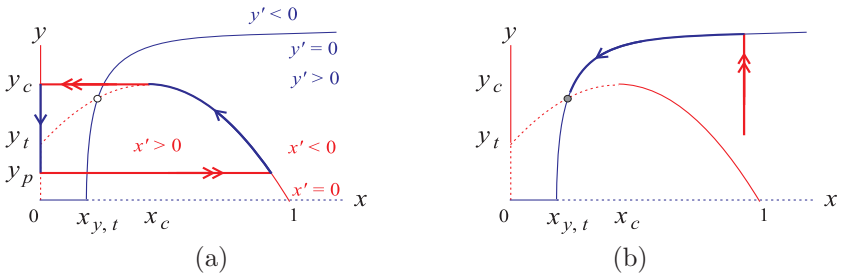


Figure 2.8

NULLCLINES

The x-nullcline consists of two branches: the trivial branch $x = 0$, and the nontrivial branch

$$y = (1 - x)(\beta + x).$$

It is a parabola with x tow intercepts: $x = -\beta$, $x = 1$. The latter $x = 1$ corresponds to the predator-free carrying capacity of the prey. The parabola has a maximal point

$$(x_c, y_c) = \left(\frac{1-\beta}{2}, \frac{(1+\beta)^2}{4}\right)$$

The case with $\beta \geq 1$ is left as an exercise. We consider here the case with

$$0 < \beta < 1$$

for which the maximal point lies in the first quadrant relevant to population dynamics.

Of the parabola, the branch right of the maximal point is the predator-mediated x-carrying capacity when each y is frozen at a fixed constant between $y = 0$ and $y = y_c$. The branch left of the maximal point is the predator mediated x-survival threshold, corresponding to unstable equilibrium point of the x-equation when y is fixed at the same constant. The intersection of the threshold branch with the trivial branch is

$$(x_t, y_t) = (0, \beta),$$

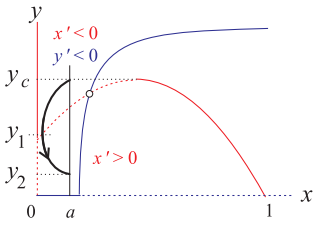
referred to as x's threshold transcritical point.

As y increases, the threshold branch increases and the capacity branch decreases in x (why, biologically?). They emerge at the critical point (x_c, y_c). Depending on the perspectives, it is a saddle-node bifurcation point of the x-equation when y is considered as parameter. It is a saddle-node turning point when both equations are considered as a singularly perturbed system for $0 < \varepsilon \ll 1$. Biologically, it is considered as a **crash-fold** point for the reason that for y lies above the level critical y_c, however slightly, the x population will crash to the extinction branch $x = 0$ regardless its initial size. Whereas for y lies below it, it is possible for the prey to stabilize at its predator mediated capacity.

Pontryagin's Delay of Loss of Stability (PDLS). Let $x = a$ be a line with $0 < a < \min\{x_{y,t}, x_c\}$. Let (a, y_c) be the initial point and $(x_\varepsilon(t), y_\varepsilon(t))$ be the solution of the perturbed system (2.13) through the initial point with $0 < \varepsilon \ll 1$. By a phase plane analysis, the solution must move down and left in the region $y' < 0$, $x' < 0$, both in a decline mode. It crosses the threshold branch vertically since at which $x' = 0$. Denote the point by $(x_1, y_1), 0 < x_1 < a, y_t < y_1$. The orbit then turns around in x direction, as the prey x is now in a recovery mode $x' > 0$, but still down for $y' < 0$. A finite time later, say $T_\varepsilon > 0$, it intersects the cross section line $x = a$ at a point denoted by $(a, y_2(\varepsilon))$.

As far as the prey population is concerned, there is no net loss during this T_ε period of time, i.e. $x_\varepsilon(0) = x_\varepsilon(T_\varepsilon) = a$, and the following expression makes this point precise:

$$0 = \frac{1}{T_\varepsilon} \ln x_\varepsilon(t)\Big|_0^{T_\varepsilon} = \frac{1}{T_\varepsilon} \int_0^{T_\varepsilon} \frac{x'_\varepsilon}{x_\varepsilon} dt.$$



This simply says that since x'/x is the per-capita growth rate of the prey, the integral represents the *average* per-capita growth rate of the prey in the period of time $[0, T_\varepsilon]$. Hence, that the average per-capita rate is zero gives the explanation as to why there is no net change of the prey population during this time.

However, there is a net decline in the predator population and that can be captured precisely by transforming the integral above in terms of the y variable as follows, using the facts that $y' \neq 0$ and $dt = dy/y'$:

$$0 = \int_0^{T_\varepsilon} \frac{\varepsilon x'_\varepsilon}{x_\varepsilon} dt = \int_{y_c}^{y_2(\varepsilon)} \frac{\varepsilon x'_\varepsilon}{x_\varepsilon y'_\varepsilon} dy_\varepsilon = \int_{y_1}^{y_2(\varepsilon)} \frac{f(x_\varepsilon, y_\varepsilon)}{y_\varepsilon g(x_\varepsilon, y_\varepsilon)} dy_\varepsilon.$$

Taking the limit $\varepsilon \to 0$ on both sides, we obtain this integral equation

$$\int_{y_p}^{y_c} \frac{f(0, s)}{s g(0, s)} ds = 0. \tag{2.14}$$

Here we have used the facts that as $\varepsilon \to 0$, $x_\varepsilon \to 0, x_1 \to 0, y_1 \to y_t = \beta$. We have also assumed that the limit $y_2(\varepsilon) \to y_p$ exists, and substituted s for y_ε as the integration variable. Referred to as the **PDLS** solution or point, y_p is such a value that prey's accumulative per-capita rate over the growing phase $y < y_t$ (i.e., $f(0, s)/(sg(0, s)) > 0$) cancels it out over the declining phase $y > y_t$ (i.e., $f(0, s)/(sg(0, s)) < 0$).

The y-nullcline has two branches as well: the trivial one $y = 0$, and the **prey-supported carrying capacity**

$$y = \frac{1}{\mu} \left(\frac{x}{\beta + x} - \delta \right). \tag{2.15}$$

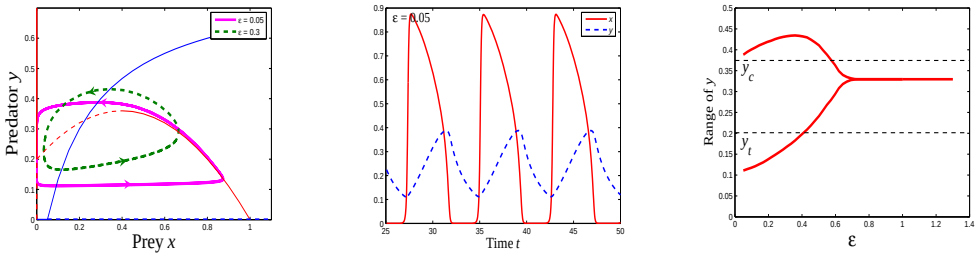
It is a monotone increasing function in x (why, biologically?), having a saturation density as the maximal capacity

$$\lim_{x \to \infty} \frac{1}{\mu} \left(\frac{x}{\beta + x} - \delta \right) = \frac{1 - \delta}{\mu}.$$

Hence as a default assumption for nontriviality we assume

$$0 < \delta < 1$$

in order for the capacity to be non-negative.



(a) (b) (c)

Figure 2.9

The two branches intersects at a capacity transcritical point

$$(x_{y,t}, y_{y,t}) = \left(\frac{\beta\delta}{1-\delta}, 0\right).$$

It is the x-intercept point of the y-capacity branch (2.15).

The combined configuration of the nullclines is such that the capacity branch of the predator intersects the threshold branch of the prey. The necessary condition for this to happen is for the y capacity transcritical value $x_{y,t}$ to lie below the crash-fold point x_c:

$$x_{y,t} = \frac{\beta\delta}{1-\delta} < \frac{1-\beta}{2} = x_c.$$

This is because the y capacity increases with increasing x, and as a result it will not intersect the x-threshold branch if its minimal point $(x_{y,t}, 0)$ already lies above the right most point, (x_c, y_c), of the threshold branch. Given this necessary condition, a sufficient condition is for μ to be small. In fact, when $\mu = 0$, the necessary condition is also sufficient because the y-capacity nullcline degenerates into a vertical line $x = x_{y,t}$, going through the threshold branch of the prey.

Hence, to summarize the nullcline configuration for the analysis below, we assume

$$x_{y,t} = \frac{\beta\delta}{1-\delta} < \frac{1-\beta}{2} = x_c \text{ and } \mu > 0 \text{ sufficient small.}$$

Singular Cycle

Figure 2.8 illustrates the two limiting cases:

(a) x is fast, y is slow when $0 < \varepsilon \ll 1$;

(b) x is slow, y is fast when $\varepsilon \gg 1$.

In (a), all singular orbits from the first quadrant converge to a limit cycle of relaxation oscillation which contains two fast orbits, initiating respectively at the crash-fold point (x_c, y_c) and the PDLS point $(0, y_p)$. In (b), all singular orbits from the first quadrant converge to the coexisting xy-equilibrium state.

Figure 2.9 are computational simulations of the system. They are generated for varying ε values but fixed values of the other parameters:

$$\beta = 0.2, \quad \delta = 0.2, \quad \mu = 1.$$

Fig.2.9(a) compares limit cycles for two different ε. The smaller ε is, the tighter the cycle is around the singular limit. Fig.2.9(b) is the time series plot for one parameter value of ε. Notice that the prey population periodically dips precariously too low. Fig.2.9(c) is a bifurcation diagram with ε being the bifurcation parameter. It is generated in the same way as the bifurcation diagram for the FitzHugh-Nagumo circuit above. Again, a nonzero variation between the minimum and maximum implies a limit cycle, and the lack of a variation means an equilibrium point. The Matlab m.file that generates the plot is `PPbifurcation.m`. Notice that the PDLS phenomenon is clearly evident. That is, the rebounding (minimum) density of the predator lies below the transcritical bifurcation value y_t. In contrast, the peak density is near the saddle-node bifurcation value y_c.

Exercises 2.5

1. Consider the equations (2.11) for the FitzHugh-Hagumo circuit. Assume the I_L-nullcline intersects the V_C-nullcline only the latter's branch left of the local minimum point. Sketch a phase portrait of singular orbits for the two singular perturbation cases: (a) $0 < \varepsilon \ll 1$, (b) $\varepsilon \gg 1$.
2. Consider the dimensionless predator-prey system (2.13) when $\beta > 1, 0 < \delta < 1$ and $x_{y,t} < 1$. Sketch a phase portrait of singular orbits for the two singular perturbation cases: (a) $0 < \varepsilon \ll 1$, (b) $\varepsilon \gg 1$.
3. Consider the dimensionless predator-prey system (2.13). In the case that $0 < \beta < 1, 0 < \delta < 1$ and $\mu = 0$, find the exact formula for the xy-equilibrium point.
4. The derivation of the PDLS point y_p from the equation (2.14) does not depends on the initial point y_c. That is, one can start at any initial point (a, y_0) with $y_0 > y_t$, and still derive the same form of the equation below for the corresponding PDLS point y_p, which now is a function of y_0, $y_p = y_p(y_0)$:
$$\int_{y_p}^{y_0} \frac{f(0,s)}{sg(0,s)} ds = 0.$$
Show that $y_p(y_0)$ is a decreasing function of y_0, and that $\lim_{y_0 \to \infty} y_p = 0$.
5. Show that the singular limit cycle of the predator-prey system (2.13) in fact contains the crash-fold point (x_c, y_c) at $\varepsilon = 0$. (*Hint:* Modify the derivation of the PDLS point y_p. More specifically, move the line $x = a$ closer to the crash-fold point so that $a < x_c$. Instead of starting at the initial point (a, y_c), start the orbit, $(x_\varepsilon(t), y_\varepsilon(t))$, at (a, y_0) for any $y_0 < y_c$. According the phase plane analysis, the orbit will cross the x-capacity nullcline at a point $(x_1(\varepsilon), y_1(\varepsilon))$, and then hit the line $x = a$ at a point, $(a, y_2(\varepsilon))$, with $y_2(\varepsilon) > y_c$. Then show $\lim_{\varepsilon \to 0} y_2(\varepsilon) = y_c$.)

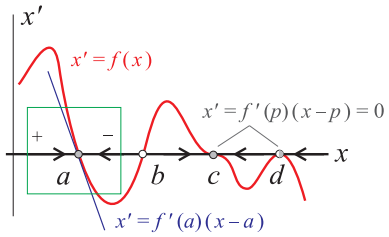
2.6 LINEARIZATION

The phase line, phase plane, and time scale methods are both geometrical and global in nature. In this section however we turn our attention to a method for the stability of equilibrium solutions that is analytical and local instead. It is based on Taylor's expansion of functions and the stability analysis for linear equations from Sec.2.1.

The basic idea can be fixed by considering autonomous equations of 1 variable

$$x' = f(x).$$

The dynamics of the equation is captured completely by the phase line method as shown by example below.



In particular, the stability of all equilibrium points can be classified qualitatively. Therefore, there is little to gain geometrically. However, if we want to know how fast solutions converge to equilibrium a, for example, the phase line method is ineffective. Since such a question only concerns a small interval of the equilibrium point a, we can simply ignore the vector field f outside the interval. If we keep zooming in at a, then the graph of the equation $x' = f(x)$ would look like a line—the tangent line of $f(x)$ at a: $x' = f(a) + f'(a)(x - a) = f'(a)(x - a)$ since $f(a) = 0$. In other words, the original equation, which is usually nonlinear, is approximated to the first order by the following linear equation

$$x' = f'(a)(x - a)$$

which is called the **linearization** of the original equation at equilibrium a.

Property of Hyperbolicity: If $f(q), f'(q) \neq 0$, then all essential information, geometrical and analytical, about the nonlinear equation near the equilibrium point q are preserved.

Take the case in the illustration that $f'(a) < 0$. Geometrically, $f(x) > 0$ if and only if $a-\delta < x < a$ for a small $\delta > 0$. Equivalently, $f'(a)(x-a) > 0$ if and only if $x < a$ for the linearization. Therefore, the phase line structures of both the nonlinear equation and its linearization near a is one of the same. Most important of all, no small perturbations can change the overall

Higher Order Approximations. The idea of approximation prevails always. Take the case in illustration as an example that is about the equilibrium point $x = d$. The linearization equation $x' = 0$ would predict that all solutions near $x = d$ are equilibrium solutions which clearly is false for the original equation. However, we can consider the approximation that is one order higher than the tangent line approximation of the vector field. More precisely, at the equilibrium point d at which the nonlinear vector field f has a quadratic-like tangency to the x-axis, the second order Taylor's approximation of the vector field is

$$f(x) \approx f(d) + f'(d)(x-d) + \frac{f''(d)}{2!}(x-d)^2 = \frac{f''(d)}{2}(x-d)^2 \neq 0$$

assuming $f(d) = f'(d) = 0$ but $f''(d) \neq 0$. Thus the following equation

$$x' = \frac{f''(d)}{2}(x-d)^2$$

is an approximation of the nonlinear equation near $x = d$. It is a good approximation for two reasons. Qualitatively it gives the same phase line structure near the point as the original equation. Analytically, one can first check or derive by the method of separation of variables the following solution to the initial value problem with $x(0) = x_0$ of the approximating equation:

$$x(t) = d + \frac{x_0 - d}{1 - \lambda(x_0 - d)t}, \text{ with } \lambda = \frac{f''(d)}{2}.$$

Secondly, one can show that the difference between the exact soluiton, $\phi(t, x_0)$, and the approximate solution is of a smaller order than $1/|t|$ as $t \to \pm\infty$.
Similarly, for $x = c$ at which f has a cubic-like tangency to the x-axis, the third order Taylor's approximation is used to derive the approximating equation

$$x' = \frac{f'''(c)}{3!}(x-c)^3,$$

assuming $f(c) = f'(c) = f''(c) = 0$ but $f'''(c) \neq 0$. Again, one can use it to analyze the original equation near $x = c$ both geometrically and analytically.
To summarize, linearization is simply a special but important case in the approximation theory of differential equations. All degenerate cases, i.e. $f(q) = f'(q) = 0$, are only secondary and exceptions to the hyperbolicity (i.e. $f'(q) \neq 0$). Unlike the hyperbolicity, they can be removed by targeted perturbations to the equations.

qualitative structures of the equilibrium point, the essence of **hyperbolicity** by definition.

Analytically, we can solve the linearization equation explicitly as follows. If we let $u = x - a$ as the new variable, then

$$u' = (x-a)' = x' = f(a)(x-a) = f'(a)u.$$

That is, the linearization equation is transformed into

$$u' = \lambda u$$

where $\lambda = f'(a)$ is a constant. Its general solution is given by

$$u(t) = Ce^{\lambda t} \iff x(t) = a + Ce^{f'(a)t}$$

Hence, for any initial point x_0 from the interval $(a - \delta, a + \delta)$, the solution with $x(0) = x_0$ is given as

$$x(t) = a + (x_0 - a)e^{f'(a)t}$$

which converges to a exponentially fast. It is a theoretical fact that this solution to the linearization approximates the solution to the original equation. The precise result can be stated as follows.

> *There is a sufficiently small $\delta > 0$ so that for any initial point x_0 from the interval of $I = (a - \delta, a + \delta)$, the solution denoted by $\phi(t, x_0)$ to initial value problem $x' = f(x), x(0) = x_0$ stays in I for all $t > 0$ and the difference between the exact solution and the linear approximation*
>
> $$|\phi(t, x_0) - [a + (x_0 - a)e^{f'(a)t}]|$$
>
> *converges to zero at a rate faster than the exponential decay $e^{f'(a)t}$ as $t \to +\infty$.*

The case in illustration that $f'(b) > 0$ is the same except that the exact and the approximate solution approach each other backward in time and at a rate faster than the exponential $e^{f'(b)t}$ as $t \to -\infty$. To be precise,

$$\lim_{t \to -\infty} \frac{|\phi(t, x_0) - [b + (x_0 - b)e^{f'(b)t}]|}{e^{f'(b)t}} = 0.$$

The linearization fails if $f' = 0$ at the equilibrium point, such as the cases in illustration for $x = c$ and $x = d$. The linearization $x' = 0$ is inconclusive about the dynamics of the nonlinear equation near these points. In such a case, higher order approximations of the vector field apply, see discussion from the inserted box.

We now extend the linearization method to 2-dimensional systems of equations

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y). \end{cases} \tag{2.16}$$

Assume $(x, y) = (p, q)$ is an equilibrium point, i.e.,

$$f(p, q) = g(p, q) = 0,$$

and we want know the local stability of the equilibrium point as well as local solution structure of the equation near the point. The principle rationale of approximation is the same: solutions of the original equations are determined by the vector field (f, g) their equations define, and as a result, an approximating vector field of the exact vector field should give approximating solutions to the exact solutions. This approach is particularly useful near equilibrium points. The approximation is again based on Taylor's approximation of functions.

As the method goes, we approximate the vector field at the equilibrium point (p, q) by their tangent planes respectively as

$$\begin{aligned} f(x,y) &\approx f(p,q) + \frac{\partial f}{\partial x}(p,q)(x-p) + \frac{\partial f}{\partial y}(p,q)(y-q) \\ &= \frac{\partial f}{\partial x}(p,q)(x-p) + \frac{\partial f}{\partial y}(p,q)(y-q) \\ g(x,y) &\approx g(p,q) + \frac{\partial g}{\partial x}(p,q)(x-p) + \frac{\partial g}{\partial y}(p,q)(y-q) \\ &= \frac{\partial g}{\partial x}(p,q)(x-p) + \frac{\partial g}{\partial y}(p,q)(y-q). \end{aligned}$$

The linear system of equations

$$\begin{cases} x' = \dfrac{\partial f}{\partial x}(p,q)(x-p) + \dfrac{\partial f}{\partial y}(p,q)(y-q) \\ y' = \dfrac{\partial g}{\partial x}(p,q)(x-p) + \dfrac{\partial g}{\partial y}(p,q)(y-q). \end{cases}$$

is called the **linearization** of the original system at the equilibrium point (p, q).

To streamline notations, we use

$$\begin{aligned} a &= \frac{\partial f}{\partial x}(p,q), \quad b = \frac{\partial f}{\partial y}(p,q) \\ c &= \frac{\partial g}{\partial x}(p,q), \quad d = \frac{\partial g}{\partial y}(p,q), \\ u &= x - p, \quad v = y - q, \end{aligned}$$

we rewrite the linearization as

$$\begin{cases} u' = au + bv \\ v' = cu + dv. \end{cases} \tag{2.17}$$

The solution structure of this linear system and that of the original system near the equilibrium point (p, q) is now a matter of interpretation between the two (2.16, 2.17).

Geometrical Properties of Linearization

First of all, the equilibrium point (p, q) of the original corresponds to the trivial solution $(0, 0)$ of the linearization. As a result, the origin of the (u, v) coordinate system is located at (p, q) on the (x, y) coordinate. Moreover, the u-axis is simply the translation of the x-axis to the point (p, q), and the same for the v-axis and y-axis parallel.

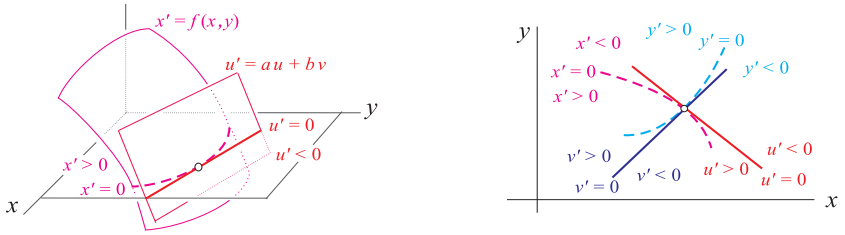
More importantly, the u-nullcline

$$u' = au + bv = \frac{\partial f}{\partial x}(p,q)(x-p) + \frac{\partial f}{\partial y}(p,q)(y-q) = 0$$

is the precisely the tangent line of the x-nullcline

$$f(x, y) = 0$$

at the equilibrium point (p, q). The most important of all is the property that both equations have the same sign polarity near the equilibrium point. This means that if the left side of the x-nullcline $f(x, y) = 0$ is where $x' > 0$ locally near the equilibrium point, then it is the left side of the u-nullcline that is where $u' > 0$. This is simply due to the fact that the u-equation is the tangent plane approximation of the x-equation when both are viewed as surfaces, see the illustration below.



It is easy to see that u' fails to capture the sign of x' near the equilibrium point if $u' \equiv 0$, that is, $f_x(p, q) = f_y(p, q) = 0$. In such a case, higher order approximation is required, which we will not pursue here.

The exactly same result and explanation apply to the y-equation and its linearization $v' = cu + dv$.

Let

$$J(p,q) = \begin{bmatrix} \frac{\partial f}{\partial x}(p,q) & \frac{\partial f}{\partial y}(p,q) \\ \frac{\partial g}{\partial x}(p,q) & \frac{\partial g}{\partial y}(p,q) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is referred to as the **Jacobian** of the vector field at point (p, q). By definition, the equilibrium point (p, q) is said to be **hyperbolic** if $J(p, q)$ does not have an eigenvalue whose real part is zero. (The real part of a real number is the real number itself.)

Topological Conjugacy: The geometrical structures of the nonlinear system and its linearization near a hyperbolic equilibrium point are the same (which has its technical definition called topological conjugacy). In particular, the equilibrium point is stable for both systems or unstable for both systems.

Stability of Hyperbolic Equilibrium Points

The linearized system (2.17) can be written in a matrix form as

$$\begin{pmatrix} u \\ v \end{pmatrix}' = J(p,q) \begin{pmatrix} u \\ v \end{pmatrix}.$$

From Sec.2.1 we know that the stability of its equilibrium point is determined by the eigenvalues of the coefficient matrix $J(p, q)$. Hence, we have the following result

---

Stability Criteria: A hyperbolic equilibrium point (p, q) of the nonlinear system (2.16) is

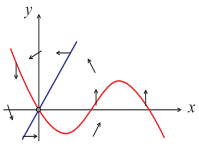
- stable if the real parts of all eigenvalues of the Jacobian matrix $J(p,q)$ are negative.
- unstable if the real part of one eigenvalue of the Jacobian matrix $J(p,q)$ is positive.

Example 2.6.1 Use linearization to determine the stability of the equilibrium point $(0,0)$ of the FitzHugh-Nagumo system

$$\begin{cases} x' = -x(x-a)(x-1) - y \\ y' = 2x - y, \end{cases}$$

where $0 < a < 1$.

Solution: Let $f(x,y) = -x(x-a)(x-1) - y$, $g(x,y) = 2x - y$. Then



$$J(0,0) = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} = \begin{bmatrix} -a & -1 \\ 2 & -1 \end{bmatrix}.$$

The linearization of the system at $(0,0)$ is

$$\begin{cases} u' = -au - v \\ v' = 2u - v. \end{cases}$$

The characteristic equation is

$$|J(0,0) - \lambda I| = \lambda^2 + (1+a)\lambda + (a+2) = 0.$$

The eigenvalues are

$$\lambda_{1,2} = \frac{-(1+a) \pm \sqrt{(1+a)^2 - 4(a+2)}}{2}.$$

Since $(1+a)^2 - 4(a+2) = (1-a)^2 - 8 < 0$ when $0 < a < 1$, the eigenvalues are complex with real part $-(1+a)/2 < 0$. Hence, the equilibrium point is stable for all $0 < a < 1$.

The example above is one of such a case that it is not too hard to find the equilibrium point and the Jacobian explicitly and the linearization method is then used directly. The following example is quite different. Finding the equilibrium and the Jacobian proves to be too tedious and too complex if not impossible when there is a simpler alternative. The key is to make use of geometrical properties of the linearization.

Example 2.6.2 Consider the dimensionless predator-prey model Eq.(2.13), which is recalled as follows.

$$\begin{cases} \varepsilon x' = x(1-x) - \dfrac{x}{\beta + x} y \\ y' = y\left(\dfrac{x}{\beta + x} - \delta - \mu y\right). \end{cases}$$

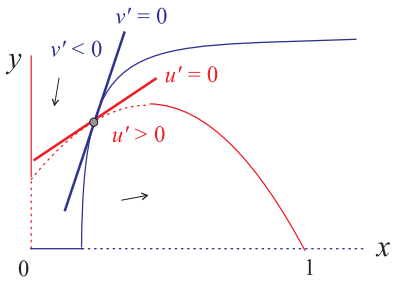
Consider also the following conditions

$$x_{y,t} = \frac{\beta\delta}{1-\delta} < \frac{1-\beta}{2} = x_c \text{ and } \mu > 0 \text{ sufficient small.}$$

so that if x is fast ($0 < \varepsilon \ll 1$) the system has a singular attracting limit cycle and if x is slow ($\varepsilon \gg 1$) the system has a singular stable equilibrium. (See Sec.2.5.) Use the method of linearization to determine how the stability of the xy-equilibrium point changes with the singular parameter ε.

Solution: Let

$$\begin{cases} \varepsilon u' = au + bv \\ v' = cu + dv \end{cases}$$



be the linearization, where $a = \partial f/\partial x$ evaluated at an/the xy-equilibrium point and so on. We already know that the u-nullcine $au+bv = 0$ is the tangent line to the x-nullcline at the equilibrium point. Since the tangent line has a positive slope we have $ab < 0$. Since we also know that $u' > 0$ for (u, v) lies below the tangent line, we must have $b < 0$ (why?). Hence we must have $a > 0$ and $b < 0$ as a result. Similarly, $cd < 0$ because the v-nullcline has a positive slope. Again using the fact that $v' < 0$ for points above it, we conclude that $c > 0, d < 0$. Notice also that the slop of the v-nullcline, $-c/d$, is greater than the slope of the u-nullcline, $-a/b$. Thus,

$$-c/d > -a/b \iff ad - bc > 0. \tag{2.18}$$

The last inequality is obtained using the fact that $b < 0, d < 0$

We now consider the eigenvalues of the Jacobian at the equilibrium point. The characteristic equation is

$$\det \begin{vmatrix} \frac{a}{\varepsilon} - \lambda & \frac{b}{\varepsilon} \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (\frac{a}{\varepsilon} + d)\lambda + \frac{ad - bc}{\varepsilon} := \lambda^2 - B\lambda + C$$

where for simpler notation we used

$$B = \frac{a}{\varepsilon} + d, \; C = \frac{ad - bc}{\varepsilon}.$$

Because $a > 0, d < 0$, $B > 0$ if $\varepsilon \ll 1$ and $B < 0$ if $\varepsilon \gg 1$. Also, because of (2.18), $C > 0$ and $C \sim +\infty$ if $\varepsilon \ll 1$ and $C \sim 0$ if $\varepsilon \gg 1$. The eigenvalues are

$$\lambda_{1,2} = \frac{B \pm \sqrt{B^2 - 4C}}{2}.$$

We now see that

- For $\varepsilon \gg 1$, $B \sim d < 0$ and $C \sim 0^+$, implying $\lambda_{1,2} < 0$. Hence the xy-equilibrium point is stable.

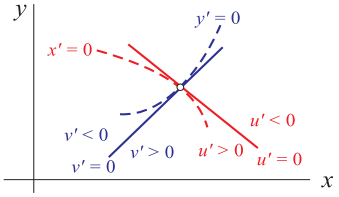
- For $\varepsilon \ll 1$, we have $B \sim +\infty$, $C \sim +\infty$, and $B^2 - 4C > 0$ but $\sqrt{B^2 - 4C} < B$. Hence, the xy-equilibrium point is a source, thus unstable.
-

In conclusion, as ε changes from very small to very large, the equilibrium point changes from being unstable to stable. This is consistent with the bifurcation plot of Fig.2.9(c) from the previous section. ◎

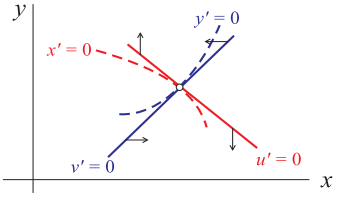
Exercises 2.6

1. Some information of the linearization of a system of equations at an equilibrium point is given in each (a) and (b). Use the information to determine the stability of the equilibrium point.

(a)



(b)



2. Verify that $x = 2$ is an equilibrium solution of each equation. Use linearization to determine its stability.

   (i) $x' = x^3 - x^2 - 2x$

   (ii) $x' = 3x - x^2 - \dfrac{3x}{1+x}$

3. Consider the system of two competing species

$$\begin{cases} x' = x(3 - 3x - y) \\ y' = y(2 - y - x). \end{cases}$$

   Find the equilibrium point that is inside the first quadrant, and use linearization to determine its stability.

4. Find the equilibrium points of the system, and use linearization to determine their stability.

$$\begin{cases} x' = 2x(1 - x^2) - y \\ y' = x - y. \end{cases}$$

5. Use linearization method to prove that the equilibrium point of Exercise 1(a) above is always stable.

6. Use linearization method to prove that the coexisting equilibrium point of the model of two competing species Eq.(2.6) is always a saddle if none of the species is competitive.