Exam 3 Solutions

1. If you use d-notation, then the partial derivatives are

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{dy}{ds},$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

2. By the chain rule,

$$\begin{aligned} w_x &= f_u\, u_x + f_v\, v_x \\ &= x f_u + y f_v. \end{aligned}$$

And by the chain and product rules,

$$\begin{aligned} w_{xx} &= f_u + x\,[f_{uu}u_x + f_{uv}v_x] + y\,[f_{vu}u_x + f_{vv}v_x] \\ &= f_u + x^2 f_{uu} + 2xy f_{uv} + y^2 f_{vv}. \end{aligned}$$

By a similar calculation,

$$w_{yy} = -f_u + x^2 f_{vv} - 2xy f_{uv} + y^2 f_{uu}.$$

Therefore,

$$w_{xx} + w_{yy} = (x^2 + y^2)\,(f_{uu} + f_{vv}) = 0.$$

3a. The linear approximation is

$$\begin{aligned} z &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 2 + 2(x-2) - (y-1) \\ &= -1 + 2x - y. \end{aligned}$$

3b. With $dx = .02$ and $dy = -.04$, we get the approximation

$$\begin{aligned} f(2.02) - f(2,1) &\approx f_x(2,1)\,dx + f_y(2,1)\,dy \\ &= .08. \end{aligned}$$

4a. The directional derivative is

$$D_{\vec{v}}g(2,1,1) = \nabla g(2,1,1) \cdot \left(\frac{\vec{v}}{|\vec{v}|}\right) = \sqrt{2}.$$

4b. The unit direction of most rapid increase is $\nabla g(2,1,1)/|\nabla g(2,1,1)| = \langle 1,2,-1\rangle/\sqrt{6}$. The derivative in that direction is $|\nabla g(2,1,1)| = \sqrt{6}$.

5. The surface is a level surface of the function $F(x,y,z) = x^3 + xz + y^2 + z^2$. Hence the normal at $(1,-1,1)$ is $\nabla F(1,-1,1) = \langle 4,-2,3\rangle$.

6a. At a critical point, $\nabla f(x, y) = \vec{0}$, that is

$$\begin{cases} 4x - 2y = 0, \\ -3y^2 - 2x = 0. \end{cases}$$

The solutions are $(0, 0)$ and $(-1/6, -1/3)$.

6b. By the second derivative test, f has a saddle at $(0, 0)$, and a local minimum at $(-1/6, -1/3)$.

7. The Lagrange multiplier method yields the three equations

$$\begin{cases} 2e^{2x+y} = 2\lambda x, \\ e^{2x+y} = 2\lambda y, \\ x^2 + y^2 = 5. \end{cases}$$

The solutions are $(2, 1)$ and $(-2, -1)$. Hence the maximum value is $f(2, 1) = e^5$, and the minimum, $f(-2, -1) = e^{-5}$.