Exam 2 Solutions

1. The plane passing through $(1, 2, -1)$ has normal $\langle 3, 1, 2\rangle$. Hence its equation is

$$3(x-1) + (y-2) + 2(z+1) = 0.$$

2a. Since gravity alone acts on the projectile,

$$\vec{r}\,''(t) = -9.8\vec{\jmath}. \tag{1}$$

The initial velocity is

$$\vec{r}\,'(0) = 200\vec{\imath} + 200\sqrt{3}\vec{\jmath}, \tag{2}$$

and the initial position,

$$\vec{r}\,(0) = \vec{0}. \tag{3}$$

Integrate twice, using (2) and (3) to evaluate the constants of integration. You'll get

$$\vec{r}\,(t) = 200t\vec{\imath} + (200\sqrt{3}t - 4.9t^2)\vec{\jmath}. \tag{4}$$

2b. The projectile is on the ground when the vertical component of $\vec{r}\,(t)$ is equal to zero. This happens at $t = 0$ (the launch time) and at $t^* = 22\sqrt{3}/4.9 \approx 70.7$ sec, (the impact time).

2c. The length of the trajectory is

$$L = \int_0^{t^*} |\vec{r}\,'(t)|\, dt = \int_0^{70.7} \sqrt{200^2 + (200\sqrt{3} - 4.9t)^2}\, dt.$$

3. The level curves of f are the concentric ellipses

$$x^2 + \frac{y^2}{4} = c, \quad c \geq 0.$$

4. On the ray $y = x$, $x > 0$, f has value

$$f(x, x) = \frac{x^2 - 1}{x^2 + 1}.$$

So as $(x, y) \to (0, 0)$ along this ray, $f(x, y) = f(x, x) \to -1$. On the parabolic arc $y = x^2$, $x > 0$, f has value

$$f(x, x^2) = 0.$$

So as $(x, y) \to (0, 0)$ along this, curve $f(x, y) = f(x, 2x) \to 0$. So by the two paths test, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$. (There is more than one pair of paths that will work.)

5a. The first partial derivatives are

$$\frac{\partial g}{\partial u} = 4v(uv + v^2 - 1)^3 \quad \text{and} \quad \frac{\partial g}{\partial v} = 4(u + 2v)(uv + v^2 - 1)^3.$$

5b. The first partial derivatives are

$$\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha) \quad \text{and} \quad \frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha).$$

6. Since $u_x = 2x\cos(x^2 - y)$ and $u_y = -\cos(x^2 - y)$, the second partial derivatives of u are

$$u_{xx} = 2\cos(x^2 - y) - 4x^2 \sin(x^2 - y), \quad u_{xy} = u_{yx} = 2x\sin(x^2 - y),$$

and

$$u_{yy} = -\sin(x^2 - y).$$

7. The equation defining z as a function of x and y is

$$z + e^{xz} + \ln(x + y) = y^2. \tag{5}$$

7a. Differentiate equation (5) with respect to y:

$$z_y + xz_y e^{xz} + \frac{1}{x + y} = 2y.$$

Therefore,

$$\frac{\partial z}{\partial y} = \frac{2y(x + y) - 1}{(x + y)(1 + xe^{xz})}. \tag{6}$$

7b. Plug $x = 0$, $y = 1$ into equation (5), and solve for $z = z(0, 1) = 0$. Now plug $x = 0$, $y = 1$ and $z = 0$ into (6) to get

$$z_y(0, 1) = 1.$$