MATH 208
Spring 2015

Name: Solutions

Exam # 2

This test is worth 100 points. Show all of your work to receive full and partial credit, making sure that all work and answers are legible. Books, notes, formula sheets, etc. are not allowed.

1. ([illegible] points) The function $f(x, y) = x^3 - 3x + y^3 - 3y$ has critical points at $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. Classify each of these critical points as a local maximum, local minimum, or saddle point. (Note: you do not need to verify that these are critical points).

$f_x = 3x^2 - 3, \quad f_y = 3y^2 - 3, \quad f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (6x)(6y) - 0^2 = 36xy$

$D(1,1) = 36 > 0, \quad f_{xx}(1,1) = 6 > 0$

$D(-1,1) = -36 < 0$

$D(1,-1) = -36 < 0$

$D(-1,-1) = 36 > 0, \quad f_{xx}(-1,-1) = -6 < 0$

f has saddle points at $(-1,1)$ and $(1,-1)$, a local min at $(1,1)$, and a local max at $(-1,-1)$.

2. (a) ([illegible] points) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = (x - 2)^2 + (y + 1)^2$ subject to the constraint $x^2 + y^2 = 20$.

Let $g(x,y) = x^2 + y^2$.

$\nabla f = \lambda \nabla g$

$2(x-2)\vec{i} + 2(y+1)\vec{j} = \lambda(2x\vec{i} + 2y\vec{j})$

$\cancel{2}(x-2) = \lambda \cancel{2}x, \quad \cancel{2}(y+1) = \lambda 2y$

$\frac{x-2}{x} = \lambda \qquad \frac{y+1}{y} = \lambda$

$\frac{x-2}{x} = \frac{y+1}{y}$

$y(x-2) = x(y+1)$

$\cancel{yx} - 2y = \cancel{xy} + x$

$g(-2y, y) = 4y^2 + y^2 = 20$

$5y^2 = 20$

$y^2 = 4$

$y = \pm 2$

critical points: $(-4, 2), (4, -2)$

$f(-4,2) = (-6)^2 + 3^2 = 36 + 9 = 45$

$f(4,-2) = 2^2 + (-1)^2 = 5$

f has a max of 45 at $(-4,2)$ and a min of 5 at $(4,-2)$ subject to the constraint.

(b) ([illegible] points) Find the maximum and minimum values of $f(x, y)$ subject to the constraint $x^2 + y^2 \le 20$.

$\nabla f = 2(x-2)\vec{i} + 2(y+1)\vec{j} = \vec{0}$

$2(x-2) = 0 \qquad 2(y+1) = 0$

$x = 2 \qquad y = -1$

f has a critical point at $(2,-1)$, which is inside the constraint.

$f_{xx}(2,1) f_{yy}(2,1) - (f_{xy}(2,1))^2 = 2(2) - 0 > 0$

$f_{xx}(2,1) = 2 > 0$

that critical point is a local min.

$f(2,-1) = 0 < 5$

the minimum of f subject to the constraint is 0 at $(2,-1)$, the maximum is 45 at $(-4,2)$.

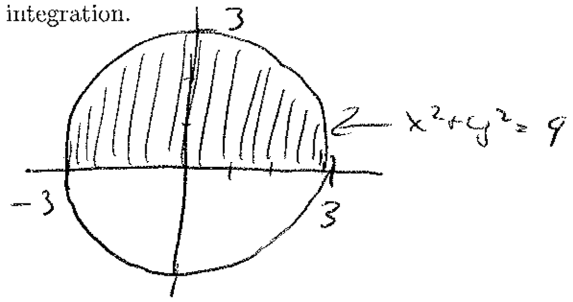
3. Consider the iterated integral: $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} 4(x^2+y^2)\,dy\,dx$.

(a) (points) Sketch the region of integration.

$y = 0$

$y = \sqrt{9-x^2}$

$y^2 + x^2 = 9$



(b) (points) Convert the integral to polar coordinates.

r goes from $r=0$ to $r=3$

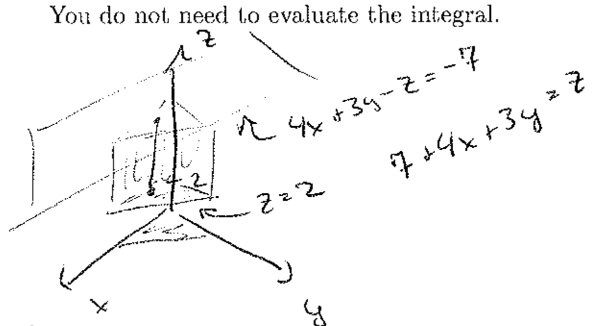
θ goes from $\theta=0$ to $\theta=\pi$

$4(x^2+y^2) = 4r^2$

$$\int_0^{\pi}\int_0^{3} 4r^3\,dr\,d\theta$$

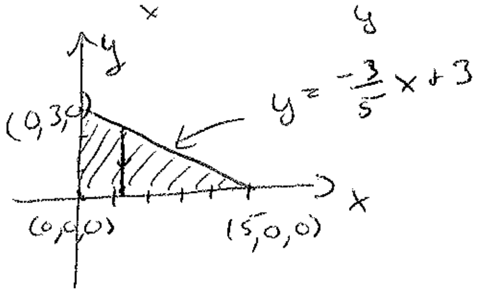
(note: $4r^2 \cdot r = 4r^3$)

4. (points) Set up a triple integral in Cartesian coordinates to calculate the volume of the region between the planes $4x+3y-z=-7$ and $z=2$, and above the triangle with vertices $(0,0,0)$, $(5,0,0)$, and $(0,3,0)$. You do not need to evaluate the integral.



$7+4x+3y=z$

$$\int_0^{5}\int_0^{-\frac{3}{5}x+3}\int_2^{7+4x+3y} dz\,dy\,dx$$



(or: $$\int_0^{3}\int_0^{-\frac{5}{3}y+5}\int_2^{7+4x+3y} dz\,dx\,dy$$ )

5. (4 points) Circle your answer. The integral: $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{16} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ calculates:

(A) The mass of an object with density $\delta = \rho^2 \sin\phi$, where the object is the region inside the sphere of radius 4, and under the cone $z = \sqrt{x^2+y^2}$.

(B) The mass of an object with density $\delta = \rho^2 \sin\phi$, where the object is the region inside the sphere of radius 4, and above the cone $z = \sqrt{x^2+y^2}$.

(C) The volume of the region inside the sphere of radius 4, and under the cone $z = \sqrt{x^2+y^2}$.

(D) The volume of the region inside the sphere of radius 4, and above the cone $z = \sqrt{x^2+y^2}$.

6. (8 points) Find a parametrization for the line segment from $(3,4,1)$ to $(1,-3,5)$.

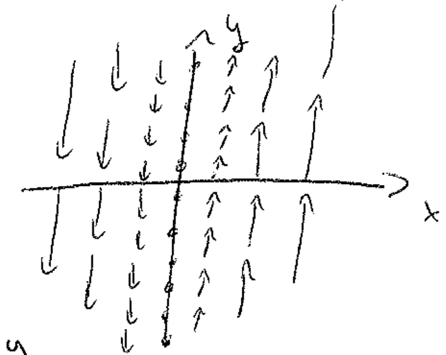
$\vec{v} = (1-3)\vec{i} + (-3-4)\vec{j} + (5-1)\vec{k}$
$= -2\vec{i} - 7\vec{j} + 4\vec{k}$

$\vec{r} = (3-2t)\vec{i} + (4-7t)\vec{j} + (1+4t)\vec{k}$
$0 \le t \le 1$

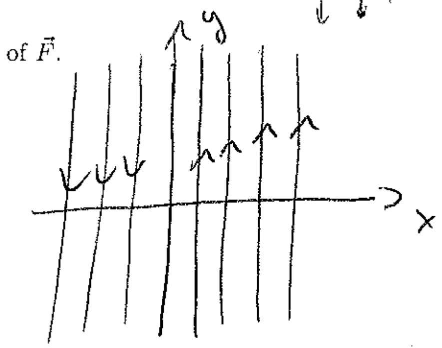
7. Consider the vector field $\vec{F} = x\vec{j}$.

(a) ([illegible] points) Sketch the vector field.

$\vec{F}(0,a) = \vec{0}$
$\vec{F}(1,a) = \vec{j}, \quad \vec{F}(2,a) = 2\vec{j}$
$\vec{F}(-1,a) = -\vec{j}, \quad \vec{F}(-2,a) = -2\vec{j}$



(b) ([illegible] points) Sketch the flow of $\vec{F}$.



(c) (8 points) Consider the parametrization $x(t) = a, y(t) = at$. Is $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ a flow line of $\vec{F}$? Show why or why not.

$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j}$
$= 0\vec{i} + a\vec{j}$
$= x\vec{j}$
$= \vec{F}$

yes, $\vec{r}(t)$ is a flow line of $\vec{F}$.

Have a good break!