MATH 208-001 Name: Solutions

Spring 2015 Exam # 1

This test is worth 100 points. Show all of your work to receive full and partial credit, using correct notation and making sure that all work and answers are legible. Books, notes, cell phones, formula sheets, etc. are not allowed.

(15 points) 1. Show the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^2}$$

Let $y = mx$

$$f(x, mx) = \frac{3x^2}{x^2+(mx)^2} = \frac{3x^2}{x^2+m^2x^2} = \frac{3x^2}{x^2(1+m^2)} = \frac{3}{1+m^2}$$

as $(x,y)\to(0,0)$ along $y=x$, $f \to \frac{3}{1+1^2} = 3/2$

as $(x,y)\to(0,0)$ along $y=2x$, $f \to \frac{3}{1+2^2} = 3/5$

$\frac{3}{2} \neq \frac{3}{5}$ so the limit does not exist

(20 points) 2. Let $\vec{v} = 2\vec{i} + 8\vec{j} - 5\vec{k}$ and $\vec{w} = 2\vec{i} + \vec{k}$.

(a) Prove that $\vec{v}$ and $\vec{w}$ are not perpendicular.

$$\vec{v}\cdot\vec{w} = 2(2) + 8(0) - 5(1) = 4-5 = -1 \neq 0$$

so $\vec{v}$ and $\vec{w}$ are not perpendicular

(b) Find the unit vector that points in the same direction as $\vec{w}$.

$$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{2^2+1^2}}\vec{w} = \frac{1}{\sqrt{5}}\vec{w} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{k}$$

(c) Find the projection of $\vec{v}$ onto $\vec{w}$, (ie. $\vec{v}_{parallel}$).

$$(\vec{u}\cdot\vec{v})\vec{u} = \left(\frac{2}{\sqrt{5}}(2) + 0(8) + \frac{1}{\sqrt{5}}(-5)\right)\vec{u} = \left(\frac{4}{\sqrt{5}} - \frac{5}{\sqrt{5}}\right)\vec{u}$$

$$= -\frac{1}{\sqrt{5}}\vec{u} = -\frac{2}{5}\vec{i} - \frac{1}{5}\vec{k}$$

(d) Find $\vec{v}_{perpendicular}$.

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = \left(2 - \left(-\frac{2}{5}\right)\right)\vec{i} + (8-0)\vec{j} + \left(-5 - \left(-\frac{1}{5}\right)\right)\vec{k}$$

$$= \left(\frac{10}{5} + \frac{2}{5}\right)\vec{i} + 8\vec{j} + \left(-\frac{25}{5} + \frac{1}{5}\right)\vec{k} = \frac{12}{5}\vec{i} + 8\vec{j} - \frac{24}{5}\vec{k}$$

3. Consider the points: $A = (2, 1, 0), B = (0, 1, 3), C = (1, 0, 1)$.

(4 points) (a) Resolve the vectors $\vec{AB}$ and $\vec{AC}$ into components.

$$\vec{AB} = (0-2)\vec{i} + (1-1)\vec{j} + (3-0)\vec{k} = -2\vec{i} + 3\vec{k}$$

$$\vec{AC} = (1-2)\vec{i} + (0-1)\vec{j} + (1-0)\vec{k} = -\vec{i} - \vec{j} + \vec{k}$$

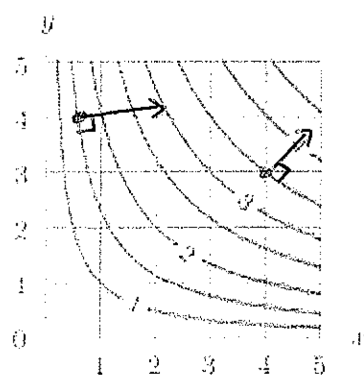
(7 points) (b) Compute $\vec{AB} \times \vec{AC}$. $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 3 \\ -1 & -1 & 1 \end{vmatrix} = (0-(-3))\vec{i} + (-3-(-2))\vec{j} + (2-0)\vec{k}$

$$= 3\vec{i} - \vec{j} + 2\vec{k}$$

(6 points) (c) Find the area of the triangle ABC. (Hint: Use the previous part.)

$$\text{area} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\sqrt{3^2 + (-1)^2 + 2^2} = \frac{1}{2}\sqrt{9+1+4} = \frac{\sqrt{14}}{2}$$

(8 points) 4. Below is a contour diagram for $f(x, y)$.



(a) Which of grad$f(0.5, 4)$ and grad$f(4, 3)$ is bigger in magnitude? grad$f(0.5, 4)$ is larger

(b) Draw grad$f(0.5, 4)$ and grad$f(4, 3)$ on the contour diagram.

(12 points) 5. Let $z = f(x,y) = e^{4y}\sin(x-y)$.

(a) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = e^{4y}\cos(x-y)$$

$$\frac{\partial z}{\partial y} = e^{4y}\cos(x-y)(-1) + e^{4y}(4)\sin(x-y)$$
$$= 4e^{4y}\sin(x-y) - e^{4y}\cos(x-y)$$

(b) Find the differential of z.

$$dz = e^{4y}\cos(x-y)\,dx + \left(4e^{4y}\sin(x-y) - e^{4y}\cos(x-y)\right)dy$$

6. Let $f(x,y) = \frac{x}{y}$.

(6 points) (a) Find $\nabla f(x,y)$.

$$\nabla f(x,y) = \frac{1}{y}\vec{i} - xy^{-2}\vec{j}$$

(10 points) (b) Compute $f_{\vec{u}}(1,1)$ where $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$. $\left(\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1\right.$ so $\vec{u}$ is a unit vector)

$$f_{\vec{u}}(1,1) = \nabla f(1,1)\cdot\vec{u}$$
$$= \left(\frac{1}{1}\vec{i} - 1(1)^{-2}\vec{j}\right)\cdot\vec{u} = (\vec{i}-\vec{j})\cdot\left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\right)$$
$$= 1\left(\frac{1}{\sqrt{2}}\right) - 1\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2}$$
$$= \sqrt{2}$$

(12 points) (c) Find the Taylor polynomial of degree two approximating $f(x,y)$ near $(2,1)$.

$$Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

$f(2,1) = \frac{2}{1} = 2$

$f_x(2,1) = \frac{1}{1} = 1$

$f_y(2,1) = -2(1)^{-2} = -2$

$f_{xx}(2,1) = 0$

$f_{xy}(x,y) = -y^{-2}$, $f_{xy}(2,1) = -1$

$f_{yy}(x,y) = 2xy^{-3}$, $f_{yy}(2,1) = 2(2)(1)^{-3} = 4$

$$Q(x,y) = 2 + 1(x-2) - 2(y-1) + \frac{0}{2}(x-2)^2 - 1(x-2)(y-1) + \frac{4}{2}(y-1)^2$$
$$= \cancel{2} + x - \cancel{2} - 2y + 2 - (x-2)(y-1) + 2(y-1)^2$$
$$= 2 + x - 2y - (x-2)(y-1) + 2(y-1)^2$$