Name: ______________________________

Circle the name of your instructor:

LUTZ	PETERSON	TRUE	KERIAN	TRUE	TRAGERSER	TRUE
8:30 am	9:30 am	10:30 am	11:30 am	12:30 pm	12:30 pm	6:30 pm

Instructions

- There are **16** questions on **9** pages (including this cover sheet).
- No books or other notes are allowed.
- You may use a calculator.
- Turn off all communication devices.
- **Show all your work** and explain your answers. Unsupported answers will receive **little credit**.
- If specified, use the method required by each problem. Alternate methods will not receive full credit.
- You have **2 hours** to complete the exam.

Good luck !

Question	Out of	Score
1	12	
2	12	
3	12	
4	12	
5	13	
6	13	
7	12	
8	12	
9	13	
10	12	
11	13	
12	12	
13	13	
14	13	
15	14	
16	12	
TOTAL	200	

1. (12 points) Given the vectors $\vec{u} = 3\vec{i} + 4\vec{k} = <3,0,4>$ and $\vec{v} = \vec{i} + 2\vec{j} - 2\vec{k} = <1,2,-2>$.

   (a) Find $\|2\vec{u} - 3\vec{v}\|$.

   $$\|(6\vec{i} + 8\vec{k}) - (3\vec{i} + 6\vec{j} - 6\vec{k})\| = \|3\vec{i} - 6\vec{j} + 14\vec{k}\|$$

   $$= \sqrt{9 + 36 + 196} = \sqrt{241}$$

   (b) Find the vector that points in the same direction as $\vec{u}$ and has length 4.

   $$\underbrace{\left(\frac{3\vec{i} + 4\vec{k}}{5}\right)}_{\text{unit}} 4 = \frac{12}{5}\vec{i} + \frac{16}{5}\vec{k}$$

   (c) Find the area of the triangle determined by $\vec{u}$ and $\vec{v}$.

   $$\frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{64 + 100 + 36} = \frac{1}{2}\sqrt{200}$$

   $$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 4 \\ 1 & 2 & -2 \end{vmatrix} = \vec{i}(-8) - \vec{j}(-10) + \vec{k}(6)$$

2. (12 points)

   (a) Find a vector normal to the surface $z^2 - 3x^2 + 4y^2 = 9$ at the point $(1,1,4)$.

   $$\vec{\nabla} g = \langle -6x, 8y, 2z \rangle$$

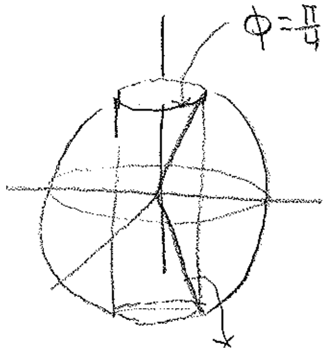
   $$\vec{n} = \vec{\nabla} g\,(1,1,4) = \langle -6, 8, 8 \rangle$$

   (b) Find an equation of the plane tangent to the above surface at $(1,1,4)$.

   $$-6(x-1) + 8(y-1) + 4(z-8) = 0$$

3. (12 points) A cylindrical drill with radius 1 is used to bore a hole through the center of a sphere of radius 2. Find a triple iterated integral in spherical coordinates (don't evaluate) that gives the volume of the ring-shaped solid that remains.

$\phi = \frac{\pi}{4}$



$\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

cone $z = r$ intersects sphere $z^2 + r^2 = 4$

when $r^2 + r^2 = 4 \Rightarrow r = \sqrt{2}$

$\Rightarrow z^2 + 2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$

$r = \sqrt{2}$ and $z = \sqrt{2}$ when $\phi = \pi/4$.

For every $0 \le \theta < 2\pi$ and $\frac{\pi}{4} \le \phi < \frac{3\pi}{4}$, ρ goes from the cylinder to the sphere

cylinder: $r = 1 \Rightarrow \rho \sin\phi = 1 \Rightarrow \rho = \frac{1}{\sin\phi}$

sphere: $\rho = 2$

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{1/\sin\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

4. (12 points) Find an equation for the tangent plane to the surface $z = f(x,y) = x^2y - 3xy^2$ corresponding to the point $(1,2)$.

$\vec{n} = \langle -f_x(1,2), -f_y(1,2), 1 \rangle$

$f_x = 2xy - 3y^2$, $f_x(1,2) = 2 - 12 = -10$

$f_y = x^2 - 6xy$, $f_y(1,2) = 1 - 12 = -11$

$\vec{n} = \langle 10, 11, 1 \rangle$

$f(1,2) = 2 - 12 = -10$

$10(x-1) + 11(y-2) + (z+10) = 0$

5. (13 points)

(a) For the function $f(x,y) = x^2 - y^2$, compute the directional derivative at $P = (1,2)$ in the direction from $P = (1,2)$ to $Q = (4,-2)$.

$$\vec{u} = \frac{3\vec{i} - 4\vec{j}}{5}$$

$$\vec{\nabla} f(1,2) = 2\vec{i} - 4\vec{j}$$

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f \cdot \vec{u} = \frac{6+16}{5} = \frac{22}{5}$$

(b) What is the value of the largest directional derivative of $f(x,y) = x^2 - y^2$ at the point $(1,2)$.

$$\|\vec{\nabla} f(1,2)\| = \sqrt{2^2+4^2} = \sqrt{20}$$

6. (13 points)

(a) Find the **critical points** of $f(x,y) = \frac{1}{3}x^3 - 16x + 3y^2 - 2y$.

$$f_x = x^2 - 16 = 0 \Rightarrow x = \pm 4$$

$$f_y = 6y - 2 = 0 \Rightarrow y = \frac{1}{3}$$

$$(4, \tfrac{1}{3}) \qquad (-4, \tfrac{1}{3})$$

(b) Use the **Second Derivative Test** to classify each of the critical points as a local maximum, local minimum, or saddle point.

	$(4, \frac{1}{3})$	$(-4, \frac{1}{3})$
$f_{xx} = 2x$	8	-8
$f_{xy} = 0$		
$f_{yy} = 6$		
$D = 12x$	48	48
	local min	local max

7. (12 points) Given $z = x\sin y + y\sin x$, $x = 2t$, $y = 1 - t^2$, use the **chain rule** to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$
$$= (\sin y + y \cos x)2$$
$$+ (x\cos y + \sin x)(-2t)$$

8. (12 points) Given that the position vector of a particle of mass is given by

$$\vec{r}(t) = < t^2 - 4t, 6t^2 - 18t + 12 > .$$

Find the velocity vector, acceleration vector, and speed of the particle of mass **all** at time $t = 1$.

$\vec{v}(t) = \langle 2t-4, 12t-18 \rangle$

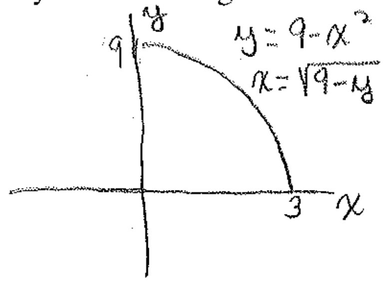
$\vec{v}(1) = \langle -2, -6 \rangle$

$\vec{a}(t) = \langle 2, 12 \rangle$

$\vec{a}(1) = \langle 2, 12 \rangle$

speed $= \sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$

9. (13 points) Sketch the region of the integration for the double integral $\int_0^3 \int_0^{9-x^2} f(x,y)\, dydx$ in the xy-plane. Clearly shade this region. Rewrite the integral reversing the order the integration.



$$\int_0^9 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$$

10. (12 points) Use **Lagrange multipliers** to find the point(s) on the ellipse $\underbrace{\frac{x^2}{2} + \frac{y^2}{8}}_{g} = 1$ where the maximum and minimum values of $f(x,y) = 2xy$ occur.

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$2y = \lambda x \qquad y = \frac{\lambda x}{2}$

$2x = \lambda \frac{y}{4} \Longrightarrow 2x = \frac{\lambda}{4}\frac{\lambda x}{2} = \frac{\lambda^2}{8}x \Longrightarrow x = 0$ or $\lambda = \pm 4$

$\frac{x^2}{2} + \frac{y^2}{8} = 1$

$x = 0$

$\frac{y^2}{8} = 1 \Rightarrow y = \pm\sqrt{8}$

$\lambda = 4$

$y = 2x$

$\frac{x^2}{2} + \frac{x^2}{2} = 1 \Rightarrow x = \pm 1$

$x = 1 \Rightarrow y = \pm 4$

$x = -1 \Rightarrow y = \pm 4$

$\lambda = -4$

$y = -2x$

$\frac{x^2}{2} + \frac{x^2}{2} = 1 \Rightarrow x = \pm 1$

same as $\lambda = 4$

Candidates

$(0, \sqrt{8})$
$(0, -\sqrt{8})$
$(1, 4)$
$(1, -4)$
$(-1, 4)$
$(-1, -4)$

$f(0, \sqrt{8}) = 0$
$f(0, -\sqrt{8}) = 0$
$\left.\begin{array}{l} f(1,4) = 8 \\ f(-1,-4) = 8 \end{array}\right\}$ max
$\left.\begin{array}{l} f(1,-4) = -8 \\ f(-1,4) = -8 \end{array}\right\}$ min

11. (13 points) Let $\vec{F}(x,y) = 2xy\vec{i} + x^2\vec{j} = < 2xy, x^2 >$ and let C be the line segment from $(0,1)$ to $(2,2)$.
(a) Find a parametrization of the curve C.

$$x = 0 + t2 \qquad dx = 2dt$$
$$y = 1 + t1 \qquad dy = dt$$
$$t: 0 \to 1$$

(b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$$\int_C 2xy\,dx + x^2 dy = \int_0^1 2(2t)(1+t)\,2dt + 4t^2 dt$$
$$= \int_0^1 (8t + 12t^2)\,dt = (4t^2 + 4t^3)\Big|_0^1 = 8$$

12. (12 points) Find the upward flux of the vector field

$$\vec{F}(x,y,z) = (z - x^2)\vec{i} + (y+1)\vec{j} + 2y\vec{k} = < z - x^2, y+1, 2y >$$

through the part of the surface $z = f(x,y) = x^2 + y^2$ above the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

$$\int_0^2 \int_0^1 \vec{F} \cdot \langle -2x, -2y, 1 \rangle \,dy\,dx$$
$$= \int_0^2 \int_0^1 (-2xz + 2x^3 - 2y^2 \cancel{-2y} \cancel{+2y})\,dy\,dx$$
$$= \int_0^2 \int_0^1 (\cancel{-2x^3} - 2xy^2 + \cancel{2x^3} - 2y^2)\,dy\,dx$$
$$= \int_0^2 \int_0^1 (-2xy^2 - 2y^2)\,dy\,dx = \int_0^2 \left(-\tfrac{2}{3}xy^3 - \tfrac{2}{3}y^3\right)\Big|_0^1 dx$$
$$= \int_0^2 \left(-\tfrac{2}{3}x - \tfrac{2}{3}\right)dx = \left(-\tfrac{1}{3}x^2 - \tfrac{2}{3}x\right)\Big|_0^2 = -\tfrac{4}{3} - \tfrac{4}{3} = -\tfrac{8}{3}$$

13. (13 points) Let Q be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above by the sphere $x^2+y^2+z^2=4$. Let S be the bounding surface of the region Q. Use the **Diverence Theorem** to find the outward flux of the vector field $\vec{F} = (xz^2)\vec{i} + \frac{1}{3}y^3\vec{j} + (x^2z+2xy)\vec{k} = < xz^2, \frac{1}{3}y^3, x^2z+2xy >$ through the surface S.

$$\operatorname{div}\vec{F} = \underbrace{z^2+y^2+x^2}$$

$$\iiint_Q \rho^2\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

sphere

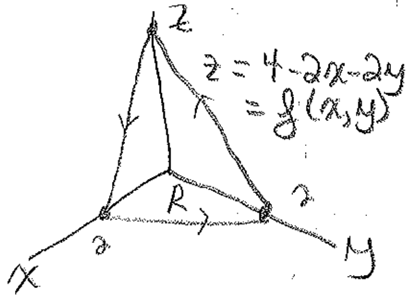
$$= \underbrace{\int_0^{2\pi}\int_0^{\pi/4}}_{\text{cone}}\int_0^{2} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

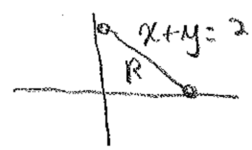
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{32}{5} \sin\phi \, d\phi \, d\theta$$

$$= -\frac{32}{5}\int_0^{2\pi} \cos\phi \Big|_0^{\pi/4} d\theta = -\frac{32}{\pi}\left(\frac{1}{\sqrt{2}}-1\right)2\pi$$

$$= 64\left(1-\frac{1}{\sqrt{2}}\right)$$

14. (13 points) Use **Stoke's Theorem** to find a **double integral** that gives the circulation of the vector field $\vec{F}(x,y,z) = z^2\vec{i} + 3xy\vec{j} + 5yz\vec{k} = < z^2, 3xy, 5yz >$ around the triangle from $P = (2,0,0)$, to $Q = (0,2,0)$, to $R = (0,0,4)$ and back to P (this triangle lies in the plane $2x+2y+z=4$). **Do not evaluate the integral.**





$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 3xy & 5yz \end{vmatrix}$$

$$= \vec{i}\, z - \vec{j}\, 2z + \vec{k}\, 3y$$

$$\iint_R < z, -2z, 3y > \cdot < \underset{-f_x}{\underset{\uparrow}{2}}, \underset{-f_y}{\underset{\uparrow}{2}}, 1 > dA$$

$$= \int_0^2 \int_0^{2-y} (2z - 4z + 3y)\, dy\, dx = \int_0^2 \int_0^{2-y} (-2\underbrace{(4-2x-2y)}_{z} + 3y)\, dy\, dx$$

15. (14 points) Let $\vec{F}(x,y) = (y+2x+1)\vec{i} + x\vec{j} = < y+2x+1, x >$ be given.

(a) First show that this vector field is a conservative vector field.

$(F_2)_x = 1 \qquad (F_1)_y = 1 \Longrightarrow$ conservative

(b) Then find a **potential** function for the given vector field.

$f_x = y + 2x + 1 \Longrightarrow f = yx + x^2 + x + \underset{y \text{ only}}{\underline{\qquad}}$

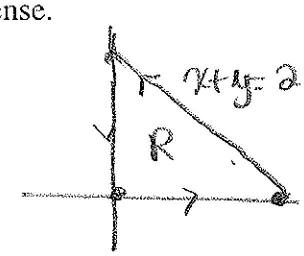
$f_y = x \Longrightarrow f = xy + \underset{x \text{ only}}{\underline{x^2 + x}}$

$f = xy + x^2 + x$

(c) Let C be the quarter of the circle $x^2 + y^2 = 1$ in the first quadrant from $(1,0)$ to $(0,1)$. Use your above answers to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = f(0,1) - f(1,0) = 0 - 2 = -2$

16. (12 points) Let $\vec{F}(x,y) = xy^2\vec{i} + 2x^2y\vec{j} = < xy^2, 2x^2y >$. Use **Green's Theorem** to evaluate the line integral $\int_C \vec{F} \cdot d\,\vec{r}$, where C is the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,2)$ oriented in the clockwise sense.



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (4xy - 2xy)\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2xy\,dy\,dx$$

$$= \int_0^2 x\,y^2\Big|_0^{2-x} dx = \int_0^2 x(2-x)^2\,dx$$

$$= \int_0^2 x(4 - 4x + x^2)\,dx$$

$$= \int_0^2 (4x - 4x^2 + x^3)\,dx$$

$$= 2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\Big|_0^2$$

$$= 8 - \frac{32}{3} + 4 = 12 - \frac{32}{3} = \frac{4}{3}$$