208 Final Exam Name: Solutions
(print)

May 5, 2009 Signature:______________________

NUID No.:______________________

Section (circle one):

Bleak (8:30)	Chouinard (9:30)	Chouinard (12:30)
Ledder (10:30)	Rebarber (11:30)	True (6:30)

INSTRUCTIONS

1. **DO NOT OPEN THIS TEST BOOKLET UNTIL INSTRUCTED TO BEGIN.**
2. Notes, books, etc. are not allowed. Calculators are allowed, but not laptop computers or equivalent, PDAs, iPods, or other similar electronics etc. and communication devices are also not allowed (this includes cell phones with calculators). **ALL CELL PHONES, PAGERS, AND SIMILAR DEVICES SHOULD BE TURNED OFF DURING THE EXAM!**
3. Note that graphs or diagrams in the questions may not be drawn in the correct proportions.
4. Simplify small integers and standard values in your answers (for example, 6 rather than $2 \bullet 3$, and ½ rather than $\cos(\pi/3)$). Answers which are complex functions or numerical values that do not easily combine need not be simplified unless the problem so states. Notation, especially in the final answers, must be correct and consistent with the text.
5. The exact answer is required for full credit on all questions, e.g. 2π rather than 6.2832.
6. Read the questions carefully, and answer them as completely as you can. (You can use the backs of pages or the bottom of page 8 if you need extra space, but indicate if you do so in the space below the problem.) **Please circle your answers.**
7. On integral problems that say DO NOT EVALUATE, all work that would need to be done prior to integration (such as dot products, finding limits, and writing everything in terms of the variables to be integrated) must be done.
8. On most questions, **supporting work is required** to get most of the credit on the problem. E.g., it's **not** acceptable to evaluate an integral on your calculator and just give the answer.
9. If a problem specifies a certain approach, little or no credit will be given for other approaches.
10. Note that this test booklet has 8 pages of questions (plus this cover sheet).
11. There is a space on each page for you to put your name or initials in case pages become separated.

Problem	Points	Score
1	17	
2	8	
3	10	
4	19	
5	9	
6	12	
7	15	
8	13	
9	12	
10	13	
11	15	
12	12	
13	15	
14	15	
15	15	
Total	200	

DO NOT OPEN THIS TEST BOOKLET UNTIL INSTRUCTED TO BEGIN.

1. Given the points $P(1,1,3)$, $Q(4,2,3)$, and $R(1,-2,2)$:

a) (5 pts) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, expressed as an inverse trig function.

$\overrightarrow{PQ} = \langle 3, 1, 0 \rangle$, $\overrightarrow{PR} = \langle 0, -3, -1 \rangle$

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = |\overrightarrow{PQ}||\overrightarrow{PR}| \cos\theta$

$-3 = \sqrt{10}\sqrt{10}\cos\theta$

$\theta = \boxed{\cos^{-1}\left(-\frac{3}{10}\right)}$

b) (4 pts) Find a non-zero vector orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ 0 & -3 & -1 \end{vmatrix} = \boxed{\langle -1, 3, -9 \rangle} \text{ works.}$$

c) (4 points) Find the area of the triangle $\triangle PQR$

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{1+9+81} = \boxed{\frac{1}{2}\sqrt{91}}$$

d) (4 pts) Write down an equation for the plane that contains P, Q, and R.

Want plane $\perp \overrightarrow{PQ} \times \overrightarrow{PR}$ and containing P.

$$\boxed{-1(x-1) + 3(y-1) - 9(z-3) = 0}$$

(or $-x + 3y - 9z = -25$, or equivalent)

2. Given the plane $3x - 2y + 5z = 34$ and the point $P(2,4,-1)$:

a) (4 points) Find a plane parallel to the given one and containing the point P.

$\langle 3, -2, 5 \rangle$ is normal

so $\boxed{3(x-2) - 2(y-4) + 5(z+1) = 0}$

(or $3x - 2y + 5z = -7$)

b) (4 points) Give parametric equations for the line passing through P and normal to the given plane.

line through P in direction $\langle 3, -2, 5 \rangle$:

$$\boxed{\begin{aligned} x &= 2 + 3t \\ y &= 4 - 2t \\ z &= -1 + 5t \end{aligned}}$$

3. (10 points) If $w = t^3 + uv^2$, $t = xe^{2y}$, $u = ye^{3x}$, and $v = x^2 y$, write down the version of the multivariable chain rule needed to find $\frac{\partial w}{\partial x}$, then use it to find $\frac{\partial w}{\partial x}$. Leave your answers in terms of *t*, *u*, *v*, *x* and *y*.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$= 3t^2 e^{2y} + v^2\, 3ye^{3x} + 2uv \cdot 2xy$$

$$= \boxed{3t^2 e^{2y} + 3v^2 ye^{3x} + 4uvxy}$$

4. Let $f(x,y,z) = y^2 - 3xy + z^2$.

a) (5 points) Find $\nabla f(1,-2,1)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle -3y, 2y-3x, 2z \rangle$$

$$\nabla f(1,-2,1) = \langle 6, -4-3, 2 \rangle = \boxed{\langle 6, -7, 2 \rangle}$$

b) (5 points) Find the rate of change of f at $\overset{P}{(1,-2,1)}$ in the direction from there toward $\overset{Q}{(4,-2,-3)}$.

$$f_{\vec{u}}\big|_P = \nabla f\big|_P \cdot \vec{u}$$

$$= \langle 6, -7, 2 \rangle \cdot \frac{\langle 3, 0, -4 \rangle}{5}$$

$$= \frac{18-8}{5} = \boxed{2}$$

$$\vec{PQ} = \langle 3, 0, -4 \rangle$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\langle 3, 0, -4 \rangle}{5}$$

c) (4 points) Find an equation for the tangent plane to the level surface $f(x,y,z) = 11$ at (1,−2,1).

plane $\perp \nabla f\big|_P$

$$\boxed{6(x-1) - 7(y+2) + 2(z-1) = 0}$$

d) (5 points) Find the unit vector in the direction of most rapid increase of f at (1,−2,1).

$$\frac{\nabla f\big|_P}{|\nabla f|_P|} = \frac{\langle 6, -7, 2 \rangle}{\sqrt{36+49+4}} = \boxed{\frac{\langle 6, -7, 2 \rangle}{\sqrt{89}}}$$

5. (9 points) Set up but **do not evaluate** an iterated integral for the flux of $\mathbf{F}(x,y,z) = 2z\mathbf{j} + xy\mathbf{k}$ over the portion of the surface given by $z = \underbrace{xy^2}_{f(x,y)}$ which satisfies $-1 \le x \le 3$ and $2 \le y \le 7$. $\overset{\shortparallel}{\langle 0, 2xy^2, xy\rangle}$

$\vec{n}\,d\sigma = \langle -f_x, -f_y, 1\rangle dy\,dx$ ← upwards

$= \langle -y^2, -2xy, 1\rangle dy\,dx$

$$\text{flux} = \int_S \vec{F}\cdot\vec{n}\,d\sigma$$

$$= \int_{-1}^{3}\int_{2}^{7} \langle 0, 2xy^2, xy\rangle \cdot \langle -y^2, -2xy, 1\rangle\, dy\,dx$$

$$= \boxed{\int_{-1}^{3}\int_{2}^{7} -4x^2y^3 + xy\, dy\,dx}$$

6. (12 points) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = xy$ on the ellipse $\underbrace{x^2 + 4y^2}_{g} = 18$.

$\nabla g = \langle 2x, 8y\rangle \neq \vec{0}$ so $\nabla f = \lambda \nabla g$ at max & min.

$\nabla f = \langle y, x\rangle = \lambda\langle 2x, 8y\rangle$

$y = 2\lambda x \Rightarrow 4y^2 = 8\lambda xy$

$x = 8\lambda y \Rightarrow x^2 = 8\lambda xy$

$4y^2 - x^2 = 0$

$x^2 = 4y^2$

$x = \pm 2y$

$x^2 + 4y^2 = (\pm 2y)^2 + 4y^2 = 18$

$8y^2 = 18$

$y^2 = \frac{9}{4}$

$x = \pm 2(\pm\frac{3}{2}) \Leftarrow y = \pm\frac{3}{2}$

$= \pm 3$

pt	f
$(3, \frac{3}{2})$	$\frac{9}{2}$
$(-3, \frac{3}{2})$	$-\frac{9}{2}$
$(3, -\frac{3}{2})$	$-\frac{9}{2}$
$(-3, -\frac{3}{2})$	$\frac{9}{2}$

min max

7. (15 points) Find and classify the critical points of $f(x,y) = 2y^3 + 6xy + x^2 + 5$.

$f_x = 6y + 2x = 0 \Rightarrow x = -3y$

$f_y = 6y^2 + 6x = 0 \Rightarrow 6y^2 - 18y = 0$

$6y(y-3) = 0$

$y = 0, 3$

CP's $(0,0)$ and $(-9,3)$

$f_{xx} = 2$ $\quad f_{xy} = 6$ $\quad f_{yy} = 12y$

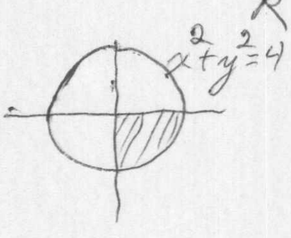
$y=0 \Rightarrow x = 0$ $\quad y = 3 \Rightarrow x = -9$

pt	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	result
$(0,0)$	2	0	6	$2\cdot 0 - 6^2 < 0$	saddle pt.
$(-9,3)$	$2>0$	36	6	$2\cdot 36 - 6^2 > 0$	local min.

8. (13 points) Set up and evaluate an expression for the mass of a lamina with density $\delta(x,y) = 4y^2$ over the region R which is the portion of the disk $x^2 + y^2 \le 4$ with $x \ge 0$ and $y \le 0$, by using an iterated integral in polar coordinates.

$r \le 2$ $\quad$ Q4, $\theta = -\frac{\pi}{2}$ + $\theta = 0$ sides.

$$m = \iint_R \delta \, dA = \int_{-\frac{\pi}{2}}^{0}\int_0^2 4(r\sin\theta)^2 \, r\,dr\,d\theta$$



$$= \left(\int_{-\frac{\pi}{2}}^{0} \sin^2\theta\,d\theta\right)\left(\int_0^2 4r^3\,dr\right)$$

$$= \int_{-\frac{\pi}{2}}^{0} \frac{1}{2}(1-\cos 2\theta)\,d\theta \; \frac{4r^4}{4}\Big|_0^2$$

$$= 16\cdot\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{-\frac{\pi}{2}}^{0}$$

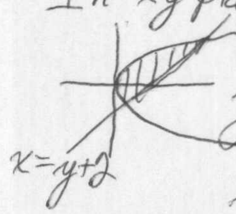
$$= 16\cdot\frac{1}{2}\left(0 - \left(-\frac{\pi}{2} - 0\right)\right) = \boxed{4\pi}$$

9. (12 points) Set up but do not evaluate a triple integral for the volume of an object bounded below by the parabolic cylinder $z = y^2$, above by the plane $z = x$, and on one side by the plane $x = y + 2$.

$$V=\int dV = \boxed{\int_{-1}^{2}\int_{y^2}^{y+2}\int_{y^2}^{x} dz\,dx\,dy}$$

$$\left(\text{or } \int_{-1}^{2}\int_{y^2}^{y+2}\int_{z}^{y+2} dx\,dz\,dy\right)$$

In xy-plane:



$x=y^2$

$y^2=y+2$

$y^2-y-2=0$

$(y-2)(y+1)=0$

$y=-1,2$

Test values

$y=0 \Rightarrow y^2=0 < y+2=2$

let $x=1$

$x=1,\ y^2=0,\ 1>0$

so all limits in correct order.

10. (13 points) Evaluate $\int_0^6 \int_{\frac{y}{2}}^3 e^{x^2}\,dx\,dy$.

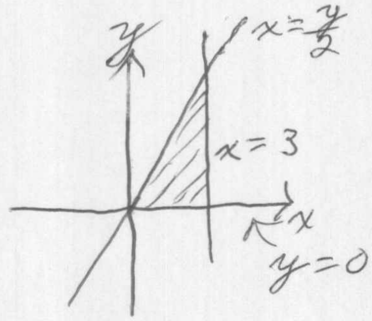
Must reverse order.

$y=2x \Leftarrow x=\frac{y}{2}$ } $\frac{y}{2}=3$

$x=3$ } $y=6$

$y=0$

~~$y=6$~~ Type 2 (intersetion)



$$\int_0^3\int_0^{2x} e^{x^2}\,dy\,dx = \int_0^3 2xe^{x^2}\,dx$$

$$= e^{x^2}\Big|_0^3 = \boxed{e^9-1}$$

11. (15 points) Convert the following to an iterated integral in spherical coordinates. **DO NOT EVALUATE.**

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} x\,dz\,dy\,dx$$

$\phi = \frac{\pi}{4}$

$\Uparrow$

$z = r \Leftarrow z = \sqrt{x^2+y^2}$

$x^2+y^2+z^2=8 \Leftarrow z = \sqrt{8-x^2-y^2}$

$\rho = \sqrt{8}$

$\Big\}\ x^2+y^2 = 8-x^2-y^2$

$x^2+y^2=4$

$y = \pm\sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$y = \sqrt{4-x^2}$ $\Big\}\ 4-x^2=0$, $x=\pm 2$

$x=0$

$\cancel{x=2}$ Type 2 (intersection)

$x=0 \Rightarrow \theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

need $x \geq 0$ so Q4 & Q1.

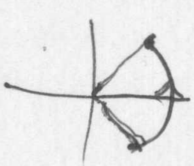
Sides are $\rho = \sqrt{8}$, $\phi = \frac{\pi}{4}$, $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$

Region has $z>0$ so $\phi \leq \frac{\pi}{4}$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \rho\sin\phi\cos\theta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

or $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$

12. (12 points) Find $\int_C x\,ds$ if C is the portion of the circle $x^2+y^2=2$ going counterclockwise from $(1,-1)$ to $(1,1)$.



$x = \sqrt{2}\cos t$

$y = \sqrt{2}\sin t$

$-\frac{\pi}{4} \xrightarrow{t} \frac{\pi}{4}$

$ds = \sqrt{(x')^2+(y')^2}\,dt$

$= \sqrt{(-\sqrt{2}\sin t)^2+(\sqrt{2}\cos t)^2}\,dt$

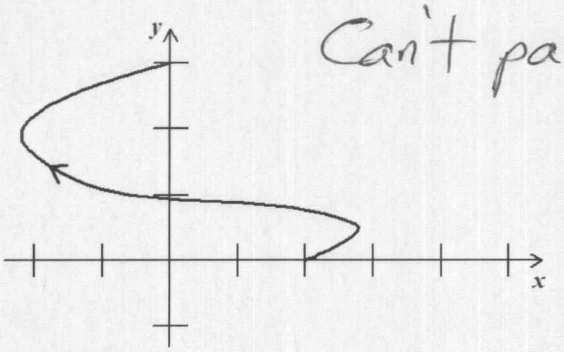
$= \sqrt{2\sin^2 t + 2\cos^2 t}\,dt$

$= \sqrt{2}\,dt$

$\displaystyle\int_C x\,ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sqrt{2}\cos t)\sqrt{2}\,dt$

$= 2\sin t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2\frac{\sqrt{2}}{2} - 2\left(-\frac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}}$

13. (15 points) Find $\int_C \mathbf{F}\bullet\mathbf{T}\,ds$ if $\mathbf{F}(x,y) = <\underbrace{y^2e^{2x}+4x}_{M}, \underbrace{ye^{2x}-1}_{N}>$ and C is the curve from (2,0) to (0,3) shown in the graph below.



Can't parametrize, is it path independent?

$N_x = 2ye^{2x} = M_y$ yes!

$f = \int y^2e^{2x} + 4x\,dx = \frac{1}{2}y^2e^{2x} + 2x^2 + \underbrace{-y}_{\text{no } x\text{'s}}$

$f = \int ye^{2x} - 1\,dy = \frac{1}{2}y^2e^{2x} - y + \underbrace{2x^2}_{\text{no } y\text{'s}}$

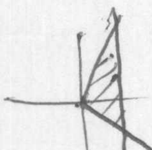
So $\int_C \vec{F}\cdot\vec{T}\,ds = \left(\frac{1}{2}y^2e^{2x} - y + 2x^2\right)\Big|_{(2,0)}^{(0,3)} = \frac{9}{2} - 3 + 0 - (0+0+8)$

$= \boxed{-\frac{13}{2}}$

14. (15 points) Use Green's Theorem to find the work done by the force $\mathbf{F}(x,y) = (\underbrace{2xy + y^3 + x}_{M})\mathbf{i} + (\underbrace{4x^2 + 3xy^2}_{N})\mathbf{j}$ on an object going once counterclockwise around the boundary of the triangle with sides $y = 2x$, $y = -x$ and $x = 4$.

$\oint_C \vec{F}\cdot d\vec{r} = \int_R N_x - M_y\,dA = \int_0^4\int_{-x}^{2x} 8x + \cancel{3y^2} - (2x + \cancel{3y^2})\,dy\,dx$

$\int_0^4 6xy\Big|_{y=-x}^{2x} dx = \int_0^4 \underbrace{12x^2 - (-6x^2)}_{18x^2}\,dx$



$= 6x^3\Big|_0^4 = \boxed{384}$

15. (15 points) Use the Divergence Theorem to find the flux of $\mathbf{F}(x,y,z) = <\underset{M}{3x}, \underset{N}{2x^2y}, \underset{P}{5z-2x^2z}>$ through the outward oriented boundary of the cylindrical region with $0 \le z \le 4$ and $x^2+y^2 \le 3$.

$$\int_S \vec{F}\cdot\vec{n}\,d\sigma = \int_D \nabla\cdot\vec{F}\,dV \qquad \nabla\cdot\vec{F} = M_x + N_y + P_z$$

$$= 3 + 2x^2 + 5 - 2x^2$$

$$= 8$$

$$= \int_D 8\,dV$$

$= 8$(volume of cylinder)

$$= 8\pi r^2 h \qquad r = \sqrt{3},\ h = 4$$

$$= 8\pi\, 3\cdot 4 = \boxed{96\pi}$$

(or you can set up the integral in cylindrical coordinates as $\int_0^{2\pi}\int_0^4\int_0^{\sqrt{3}} 8r\,dr\,dz\,d\theta$ and evaluate)