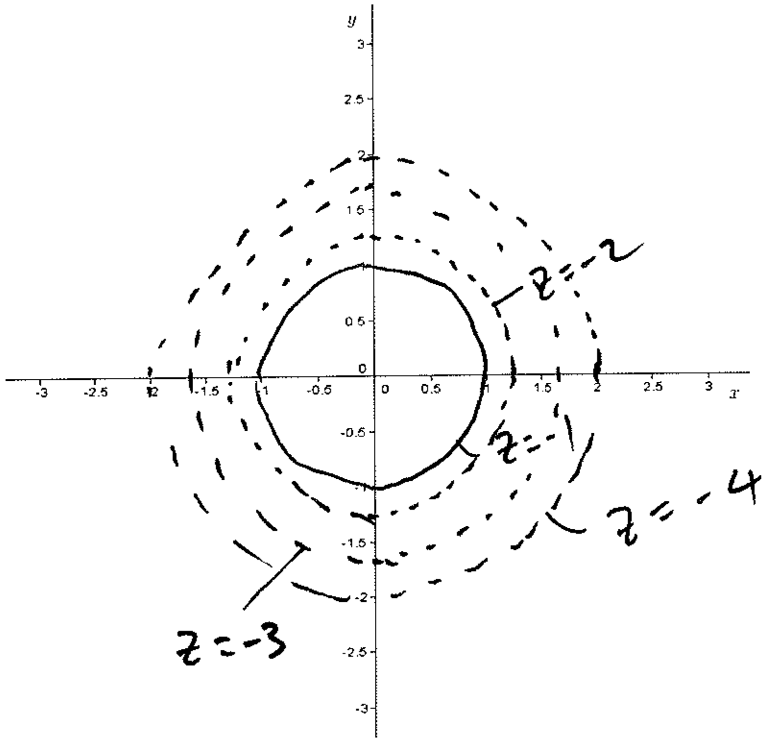
NAME: Soln. Key

Professor's name:__________ Lecture start time:__________

This exam should have 10 pages; please check that it does. Show all work that you want considered for grading. Calculators are not allowed. **An answer will only be counted if it is supported by all the work necessary to get that answer.** Simplify as much as possible, except as noted: for example, don't write $\cos(\pi/4)$ when you can write $\sqrt{2}/2$. Also, **give exact answers only**, except as noted; for example, don't write 3.1415 for π. If there are multiple parts to a problem, the parts are not necessarily worth an equal amount of points.

Problem	Maximum	
1	7	
2	12	
3	12	
4	13	
5	16	
6	14	
7	10	
8	15	
9	10	
10	12	
11	10	
12	10	
13	17	
14	10	
15	15	
16	17	
Total	200	

1. (7 points) Let $z = f(x, y) = -x^2 - y^2$. Draw level curves for $z = -1, -2, -3$ and -4 on the axes below, making sure to label each level curve.



$f(x,y) = -x^2 - y^2$
$= z = -1$
$\Rightarrow x^2 + y^2 = 1$

2. (12 points). Suppose $z = f(x, y)$, $x = x(t, s)$ and $y = y(t, s)$.

(a) Write down the chain rule for $\dfrac{\partial z}{\partial s}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= f_x(x,y)\frac{\partial x}{\partial s} + f_y(x,y)\frac{\partial y}{\partial s}$$

(b) Find $\dfrac{\partial z}{\partial s}$ when $t = 0$ and $s = 6$ if

$f_x(3,2) = 1$, $f_y(3,2) = 5$, $f_x(0,6) = -2$, $f_y(0,6) = 4$
$x(0,6) = 3$, $y(0,6) = 2$, $x_t(0,6) = -4$, $x_s(0,6) = 10$, $y_t(0,6) = -2$, $y_s(0,6) = -3$.

$$\frac{\partial z}{\partial s}(0,6) = f_x(x(0,6), y(0,6))\frac{\partial x}{\partial s}(0,6) + f_y(x(0,6), y(0,6))\frac{\partial y}{\partial s}(0,6)$$
$$= f_x(3,2)\cdot 10 + f_y(3,2)\cdot(-3)$$
$$= 1\cdot 10 + 5(-3) = -5$$

3. (12 points) Consider the plane $5x - y + 7z = 21$.

   (a) Find a point on the x-axis on this plane.

$$x = \frac{21}{5}$$

   (b) Find a vector perpendicular to the plane.

$$\vec{n} = (5, -1, 7).$$

   (c) Is the vector $\vec{i} - 2\vec{j} - \vec{k}$ parallel to the plane? Explain your answer.

$$\vec{n} \cdot (i - 2j - k)$$
$$= (5, -1, 7) \cdot (1, -2, -1) = 5 + 2 - 7 = 0$$

Yes.

4. (13 points) Let $f(x, y) = e^{x^2+y}$.

   (a) Compute grad f.

$$\text{grad } f = f_x \vec{i} + f_y \vec{j}$$
$$= (2xe^{x^2+y}, \; e^{x^2+y} \cdot 1).$$

   (b) Find $f_{\vec{u}}(1, -1)$, which is the directional derivative of f in the direction of $\vec{i} + 2\vec{j}$ at $(1, -1)$.

$$f_{\vec{u}}(1, -1) = \text{graf}(1, -1) \cdot \frac{i + 2j}{\| i + 2j \|} = \vec{u}$$
$$= (2, 1) \cdot \frac{(1, 2)}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

   (c) In what direction is the maximum directional derivative of f at the point $(1, -1)$? What is that maximum directional derivative?

$$\vec{n} = \frac{\text{grad } f}{\|\text{grad } f\|} = \frac{(2, 1)}{\sqrt{5}}$$

$$f_{\vec{n}}(1, -1) = \|\text{grad } f\| = \sqrt{5}.$$

5. (16 points) Find the critical points of $f(x,y) = xy + x^2 + (y^3/3)$, and classify each of them as a local maximum, local minimum or saddle point.

Solve: $\text{grad } f = \vec{0}$

or $\begin{cases} \dfrac{\partial f}{\partial x} = y + 2x = 0 & (1) \\ \dfrac{\partial f}{\partial y} = x + y^2 = 0. & (2) \end{cases}$

$(1) \Rightarrow \quad y = -2x \quad \Rightarrow \quad x + (-2x)^2 = 0 \Rightarrow x(1+4x) = 0$

$\Rightarrow x = 0$ and $x = -\frac{1}{4}$

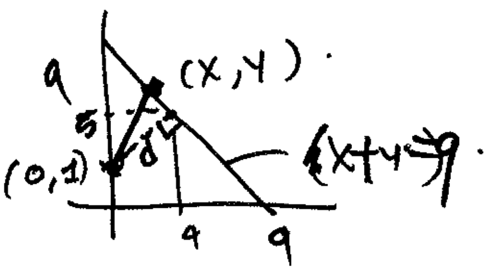
$\Rightarrow$ c.pt. $(0,0)$, $(-\frac{1}{4}, +\frac{1}{2})$

$f_{xx} = 2 > 0, \quad f_{xy} = 1, \quad f_{yy} = 2y.$

$D = f_{xx} f_{yy} - f_{xy}^2 = 2(2y) - 1^2 = 4y - 1$

	f_{xx}	D	
$(0,0)$	>0	$-1<0$	saddle
$(-\frac{1}{4}, +\frac{1}{2})$	>0	$4\frac{1}{2} - 1 = 1 > 0$	local min.

6. (14 points) Using Lagrange multipliers, determine the shortest distance from the line $x + y = 9$ to the point $(0, 1)$. Hint: minimize the square of the distance from (x, y) to $(0, 1)$. No credit will be given for any method besides Lagrange multipliers.



$\min.\ z = f(x,y) = d^2 = (x-0)^2 + (y-1)^2$

$f(x,y) = x^2 + (y-1)^2$

sub. to. $g = x + y = 9$

Solve: $\begin{cases} \text{grad} f = \lambda \, \text{grad}\, g \\ g = 9. \end{cases}$ must

$\Rightarrow \begin{cases} \overset{f_x}{2x} = \lambda \cdot \overset{g_x}{1} & ① \\ 2(y-1) = \lambda \cdot 1 \quad g_y. & ② \\ x + y = 9. & ③ \end{cases}$

① & ② : $\cancel{2}x = \cancel{2}(y-1)$, $\quad x = y - 1$.

③ $\Rightarrow (y-1) + y = 9. \Rightarrow 2y = 10,\ y = 5,\ x = 4.$

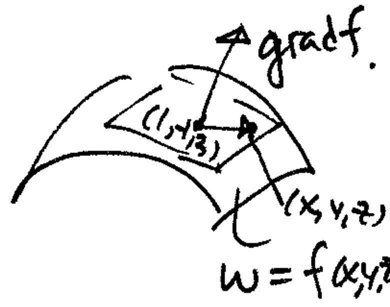
$(x, y) = (4, 5)$.

7. (10 points)

(a) Find a normal vector to the surface $x^3 + 2yz = -5$ at $(1, -1, 3)$.

f

$$\vec{n} = \operatorname{grad} f\,(1,-1,3).$$
$$= (f_x, f_y, f_z)$$
$$= (3x^2, 2z, 2y)$$
$$\vec{n} = (3, 6, -2)$$

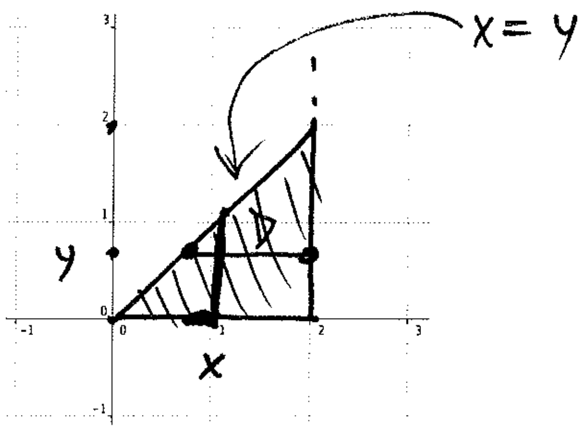


(b) Find an equation for the tangent plane to the surface $x^3 + 2yz = -5$ at $(1, -1, 3)$.

$$\vec{n} \cdot (x-1, y+1, z-3) = 0$$
$$3(x-1) + 6(y+1) - 2(z-3) = 0. \Rightarrow \ldots$$

8. (15 points) Consider the integral $\int_0^2 \int_y^2 \frac{ye^{x^2}}{x}\, dx\, dy$.

(a) Sketch the region.



(b) Change the order of integration to $dy\, dx$, and evaluate this integral.

$$\int_0^2 \int_y^2 f\, dx\, dy = \iint_D f\, dA$$
$$= \int_0^2 \int_0^x \frac{ye^{x^2}}{x}\, dy\, dx = \int_0^2 \frac{1}{2} y^2 \frac{e^{x^2}}{x} \Bigg|_{y=0}^{y=x} dx$$
$$= \int_0^2 \frac{1}{2} x^2 \frac{e^{x^2}}{x}\, dx = \frac{1}{2}\int_0^2 \underline{\underline{x e^{x^2}}}\, dx = \frac{1}{2}\left(\frac{1}{2} e^{x^2}\right)\Big|_0^2$$
$$= \frac{1}{4}\left(e^4 - 1\right)$$

9. (10 points) Let S be the closed cylindrical surface described as follows: The sides of the cylinder are $x^2 + y^2 = 4$ with $0 \le z \le 3$, and the cylinder is closed off by a top disk in the plane $z = 3$ and a bottom disk in the plane $z = 0$. Use the divergence theorem to calculate the outward flux of the vector field

$$\vec{F} = e^{y^2} \cos z\,\vec{i} + 2y\,\vec{j} - x^3 y\,\vec{k}$$

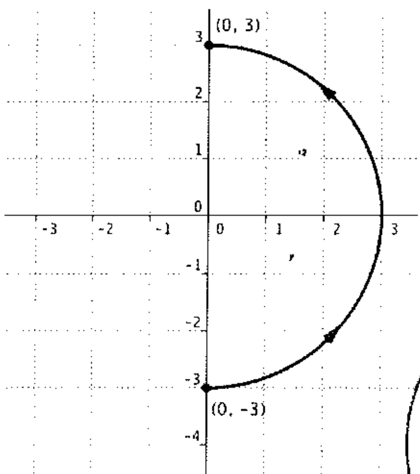
through S.

$$\int_S \vec{F}\cdot d\vec{A} \overset{\text{div. thm}}{=} \iiint_G \operatorname{div}\vec{F}\,dV = \iiint_G (0+2+0)\,dV$$

$$= 2\iiint_G dV = 2 \text{ vol of cylinder}$$

$$= 2\,\pi\,2^2\cdot 3 = 24\pi.$$

10. (12 points) Let $\vec{F} = x\vec{i} + z\vec{j} - y\vec{k}$. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the half circle of radius 3 in the yz plane, centered at the origin, oriented counterclockwise when viewed from the positive x-axis, and going from $(0,0,-3)$ to $(0,0,3)$ - see the picture, which is in the y-z plane.



$$\vec{r}(t) = (\overset{x}{0}, \overset{y}{3\cos t}, \overset{z}{3\sin t})$$

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

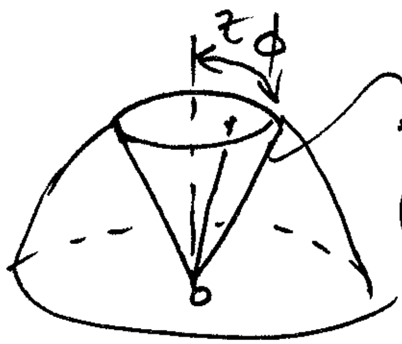
$$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{F}(0, 3\cos t, 3\sin t)\cdot \vec{r}\,'(t)\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0, 3\sin t, -3\cos t)\cdot(0, -3\sin t, 3\cos t)\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -9\sin^2 t - 9\cos^2 t\,dt = -9\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = -9\pi.$$

11. (10 points) Let W be the "ice-cream cone" solid above the cone $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 9$. Its density is given by $\delta(z) = z$, where the units for the solid are centimeters cubed, and the density is in grams per centimeter cubed. Set up a triple integral in **spherical coordinates** that gives the mass of the cone. **DO NOT EVALUATE.**



$z = \sqrt{x^2+y^2}$

$\Rightarrow \phi = \frac{\pi}{4}$

$\left(\tan\phi = \frac{z}{\sqrt{x^2+y^2}} = 1. \right)$

$$V = \iiint_G dV$$

$$\underset{sp.}{=} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

12. (10 points) Let the position of a particle be given by

$$x(t) = (1/4)t^4 - (1/2)t^2 \qquad \text{and} \qquad y(t) = 3t^2 - 6t,$$

where x and y are in centimeters and t is in seconds.

(a) Find the velocity of the particle. Give the units.

$\vec{r}\,'(t) = (t^3 - t,\ 6t - 6)$ cm/sec.

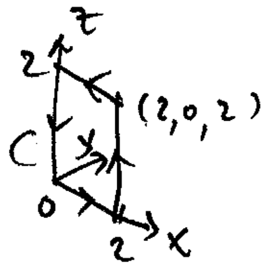
(b) Find any times when the particle stops.

$\vec{r}\,'(t) = 0, \quad t^3 - t = t(t^2 - 1) = 0, \quad t = 0, \pm 1$

and $6t - 6 = 6(t-1) = 0, \Rightarrow t = 1.$

$\Rightarrow t = 1$

13. (17 pts) Use Stokes' theorem to calculate the circulation of the vector field $\vec{F} = y\,\vec{i} + 2x\,\vec{k}$ around the rectangle that goes from $(0,0,0)$ to $(2,0,0)$ to $(2,0,2)$ to $(0,0,2)$, and back to $(0,0,0)$.



$$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}\vec{F}\cdot d\vec{A}$$

$$= \int_S (-2j - k)\cdot d\vec{A}$$

$$= (-2j - k)\cdot 4(-j)$$

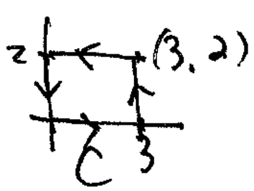
$$= 8$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 2x \end{vmatrix}$$

$$= -2j - k$$

$S:\ \vec{A} = -4j$

14. (10 points) Let $\vec{F} = (xy)\vec{i} + (x^2+xy)\vec{j}$. Consider the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the square with vertices $(0,0)$, $(3,0)$, $(3,2)$ and $(0,2)$, oriented counterclockwise. Use Green's theorem to evaluate this integral.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$= \iint_D 2x + y - x \; dA = \int_0^3 \int_0^2 x + y \, dy\, dx$$

$$= \int_0^3 xy + \tfrac{1}{2}y^2 \Big|_0^2 dx = \int_0^3 2x + 2\, dx$$

$$= x^2 + 2x \Big|_0^3 = 9 + 6 = 15$$

15. (15 points) Let $\vec{F} = (3x^2y + 2)\vec{i} + (x^3 - 1)\vec{j}$.

(a) Find a potential function for $\vec{F}$.

$$\frac{\partial f}{\partial x} = 3x^2y + 2 \quad \Rightarrow \quad f = \int 3x^2y + 2\,dx = x^3y + 2x + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 - 1 \quad \frac{\partial f}{\partial y} = x^3 + g'(y) = x^3 - 1, \quad g'(y) = -1$$

$$g(y) = -y \quad \Rightarrow \quad f(x,y) = x^3y + 2x - y$$

(b) Use the potential function to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is part of the circle of radius 2 from $(2,0)$ to $(0,2)$ (that is, the part of the circle in the first quadrant.)

$$\int_C \vec{F} \cdot d\vec{r} = f(0,2) - f(2,0) = -2 - 4 = -6$$

16. (17 points) Let S be part of $z = 4 - x^2 - y^2$ which is above the xy plane, oriented with upwards normals, and let $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$.

Set up an iterated integral in polar coordinates for the flux of $\vec{F}$ over S. Simplify the integrand as much as possible, making sure that the variables in the integrand match the variables of integration, but **DO NOT EVALUATE**.

$$\int_S \vec{F} \cdot d\vec{A} = \iint_D (x, y, 4-x^2-y^2) \cdot (-f_x, -f_y, 1)\, dA$$

$S: z = f(x,y) = 4 - x^2 - y^2$

over ~~D~~: $x^2 + y^2 \leq 4$

$$= \iint_D (x, y, 4-x^2-y^2) \cdot (2x, 2y, 1)\, dA$$

$$= \iint_D 2x^2 + 2y^2 + 4 - x^2 - y^2\, dA = \iint_D (4 + x^2 + y^2)\, dA$$

$$= \int_0^{2\pi}\int_0^2 (4 + r^2)\, r\, dr\, d\theta$$