Math107 Fall 2011
Calculus II University of Nebraska-Lincoln

Hint

There are several ways to approximate approximate the integral in the formula for T. For example, it is also possible to use the Trapezoidal Rule, but this would need to be done very carefully to get a meaningful result. The suggestion outlined here is to use Taylor polynomials to approximate some of the terms in the integrand for the formula of T. This leads to an integral that is more easily evaluated.

Problem 4

For this problem, you are required to estimate the time it takes the bead to travel from the point $(0, 0)$ to the point $(1, -2)$ by following the graph of $y = -2x^{\frac{1}{2}}$. Here are some steps you can follow to approach this problem.

1. Show that the time of travel for the bead is

$$T = \frac{1}{2\sqrt{g}} \int_0^1 \frac{1}{x^{\frac{3}{4}}} (x+1)^{\frac{1}{2}}\, dx.$$

   You should provide some work for this step in your write-up.

We want to estimate the integral above and get an upper bound for our error. One approach is to expand the term $(x+1)^{\frac{1}{2}}$ in a Taylor series centered at $x = 0$ (see Sections 8.8 and 8.9 in the text).

2. Decide how many terms you want to use in your series, let's say $n+1$, and find the Taylor polynomial of order n with remainder for $f(x) = (x+1)^{\frac{1}{2}}$:

$$(1+x)^{\frac{1}{2}} = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x).$$

3. To estimate T, plug the series (without the remainder term) in and compute:

$$T \approx \frac{1}{2\sqrt{g}} \int_0^1 \frac{1}{x^{\frac{3}{4}}} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k\, dx.$$

   This is an integral that you can evaluate exactly.

Now, what about the error? The only thing we left off was the remainder term; we'll use it to get an upper bound on the error.

$$\text{error} = \frac{1}{2\sqrt{g}} \int_0^1 \frac{1}{x^{\frac{3}{4}}} R_n(x)\, dx. \tag{1}$$

4. Explain why

$$|\text{error}| \le \frac{1}{2\sqrt{g}} \int_0^1 \frac{f^{(n+1)}(1)}{(n+1)!} x^{n+\frac{1}{4}}\, dx.$$

Thus the upper bound for the error is

$$\frac{1}{2\sqrt{g}} \frac{f^{(n+1)}(1)}{(n+1)!(n+\frac{5}{4})}$$

An Example

We'll estimate T using a series with 1 term; i.e. $n = 0$. The series is

$$(x+1)^{\frac{1}{2}} = f^{(0)}(0) + R_0(x) = 1 + R_0(x).$$

Thus

$$T \approx \frac{1}{2\sqrt{g}} \int_0^1 x^{-\frac{3}{4}}\, dx = \frac{2}{\sqrt{g}}. \tag{2}$$

Assuming that you have completed step 4 above, we get the following upper bound on the error:

$$\frac{1}{5\sqrt{g}} \tag{3}$$

What this means is that we are guaranteed that

$$\frac{2}{\sqrt{g}} - \frac{1}{5\sqrt{g}} \leq T \leq \frac{2}{\sqrt{g}} + \frac{1}{5\sqrt{g}};$$

i.e.

$$\frac{1}{\sqrt{g}} \cdot \frac{9}{5} \leq T \leq \frac{1}{\sqrt{g}} \cdot \frac{11}{5}.$$

Of course, you will need to use more terms in the series to get an estimate (and an upper bound on the error) that is useful to make comparisons.

Problem 5

For this problem, you are required to estimate the time it takes the bead to travel from the point $(0,0)$ to the point $(1,-2)$ by following the graph of $y = -2x^{\frac{3}{5}}$. This can be done by modifying the idea suggested for Problem 4. First show that the time of travel for the bead is

$$T = \frac{1}{2\sqrt{g}} \int_0^1 \frac{1}{x^{\frac{7}{10}}} \left(x^{\frac{4}{5}} + \left(\frac{6}{5}\right)^2 \right)^{\frac{1}{2}} dx.$$

To estimate this integral, we approximate the term $\left(x^{\frac{4}{5}} + \left(\frac{6}{5}\right)^2 \right)^{\frac{1}{2}}$. To do this, you can follow the steps suggested for Problem 4, but the derivatives will get pretty messy. Alternatively, you can adapt a technique illustrated in Example 5 of Section 8.9:

1. Find the Taylor polynomial $P_n(u)$ of order n for the function $f(u) = \left(u + \left(\frac{6}{5}\right)^2 \right)^{\frac{1}{2}}$.
2. The term $\left(x^{\frac{4}{5}} + \left(\frac{6}{5}\right)^2 \right)^{\frac{1}{2}}$ can be approximated with $P_n(x^{\frac{4}{5}})$, which will yield an integral that is straightforward to evaluate.
3. As was done for Problem 4, an upper bound on the error can be established using the remainder term.