Math107
Calculus II

Spring 2008
University of Nebraska-Lincoln

Name: ______________________________ **Section #:** ____________

(print legibly)

Exam 3

Instructions

- Turn off all communication devices. If you do not do so, then you will not receive any credit for your exam.
- There are 7 pages in this exam with 6 problems. Before you begin, make sure that your exam has all 7 pages.
- The examination period is from 9:30am to 10:20am. If you wish to receive credit for your exam, then make sure that your exam is submitted for grading by 10:20am.
- You may NOT use a calculator during the exam.
- You may not use a text, notes, nor any other reference.
- To receive full credit for a problem, you must provide a correct answer and a sufficient amount of work so that it can be determined how you arrived at your answer.
- Clearly indicate what your solutions are and any work that you do not want to be included in the grading process.
- Write your solutions in an explicit form whenever possible.
- If you wish to speak with a proctor during the exam, then raise your hand and a proctor will come to you.
- Each problem will be graded out of 20 points.
- If it is determined that you have given or received any unauthorized aid during this exam, then you will receive no credit for your exam.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

1. (a) Convert the equation $r = 2\sin\theta$ from polar coordinates to Cartesian coordinates.

Solution:
Multiplying the equation by r yields

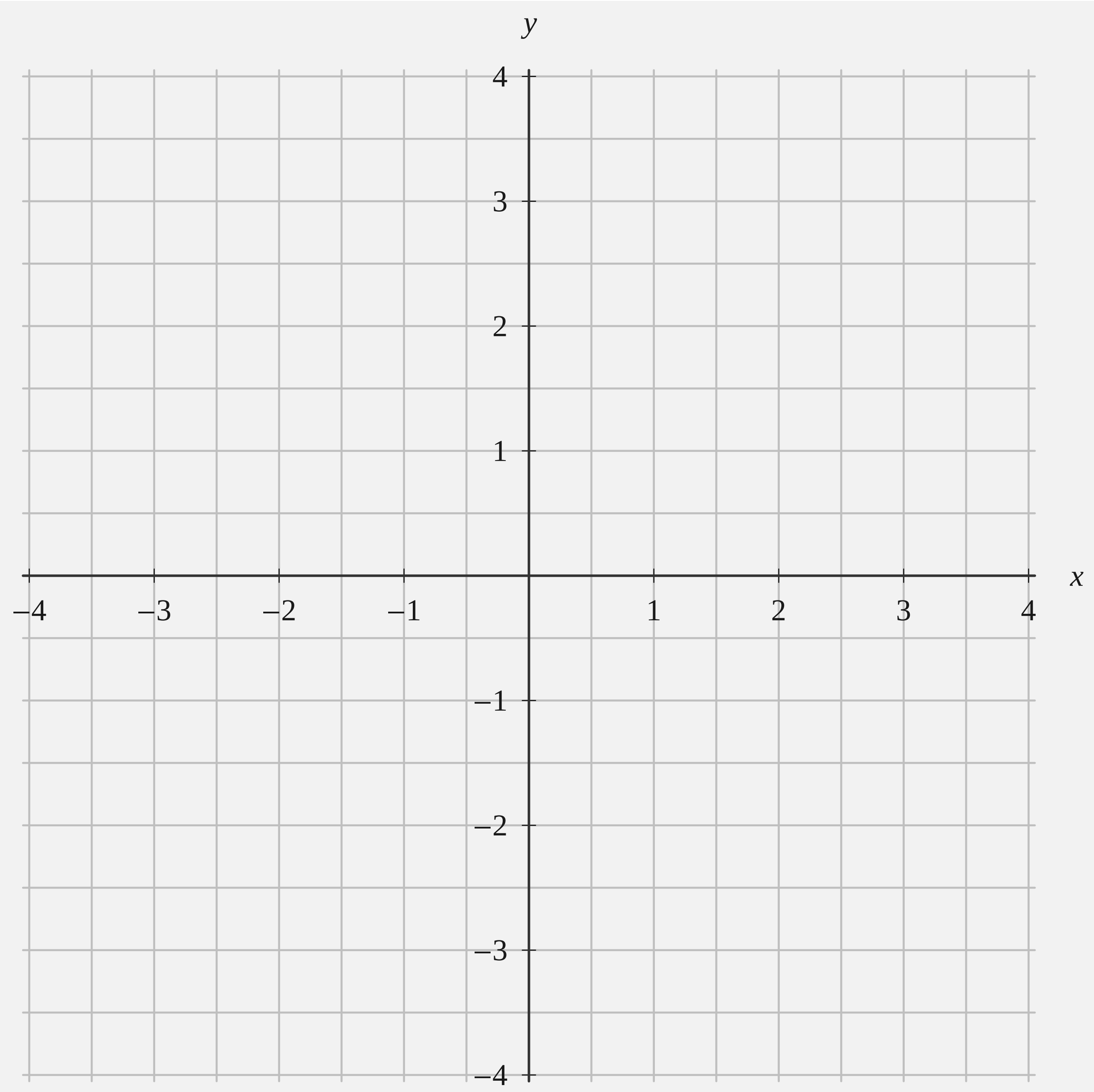
$$r^2 = 2r\sin\theta.$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, we conclude that

$$\boxed{x^2 + y^2 = 2y.}$$

(b) On the set of axes below, provide a careful graph for the set of points satisfying

$$-1 \le r \le 3 \qquad \text{and} \qquad \frac{\pi}{4} < \theta \le \frac{3\pi}{4}.$$



2. For the following problems, $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}$ is the vector with initial point $P(0,4,5)$ and terminal point $Q(-3,6,5)$.

(a) Write the vector $\mathbf{v}$ in component form.

Solution:

$$\boxed{\mathbf{v} = \langle -3, 2, 0 \rangle.}$$

(b) Find the coordinates for the point R if $\overrightarrow{QR} = 2\mathbf{u} - \mathbf{v}$.

Solution:
We see that

$$2\mathbf{u} - \mathbf{v} = 2\langle 6, 3, -2 \rangle - \langle -3, 2, 0 \rangle = \langle 9, 8, -4 \rangle.$$

Hence

$$R = (-3+9, 6+8, 5-4) = \boxed{(6, 14, 1).}$$

(c) Find a vector that is 10 units in length and has the same direction as $\mathbf{u}$.

Solution:
The length of $\mathbf{u}$ is

$$|\mathbf{u}| = \sqrt{6^2 + 3^2 + (-2)^2} = \sqrt{49} = 7.$$

Thus

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \langle \frac{6}{7}, \frac{3}{7}, -\frac{2}{7} \rangle$$

is a unit vector with the same direction as $\mathbf{u}$. A vector with length 10 and the same direction as $\mathbf{u}$ is

$$10\frac{\mathbf{u}}{|\mathbf{u}|} = \boxed{\langle \frac{60}{7}, \frac{30}{7}, -\frac{20}{7} \rangle.}$$

3. For the following problems, $\mathbf{u} = \langle 1, -1, 0\rangle$ and $\mathbf{v} = \langle 0, 2, 2\rangle$

(a) Compute $\mathbf{u} \cdot \mathbf{v}$

Solution:
Using the definition of the dot product,

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 - 1 \cdot 2 + 0 \cdot 2 = \boxed{\text{-2.}}$$

(b) What is the exact angle between $\mathbf{u}$ and $\mathbf{v}$? EXPRESS YOUR ANSWER WITHOUT INVERSE TRIG FUNCTIONS.

Solution:
We have that $\mathbf{u} \cdot \mathbf{v} = -2$. Also

$$|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

and

$$|\mathbf{v}| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8}.$$

It follows that the angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\cos^{-1}\left(\frac{-2}{\sqrt{2}\sqrt{8}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3} \text{ radians.}}$$

(c) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

Solution:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{-2}{8}\langle 0, 2, 2\rangle = \boxed{\langle 0, -\frac{1}{2}, -\frac{1}{2}\rangle.}$$

4. For the following problems,

$$\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{t+2}\mathbf{j} - e^{2t}\mathbf{k}.$$

(a) At what values of t is $\mathbf{r}(t)$ continuous?

Solution:
The first and third components of $\mathbf{r}$ are continuous everywhere. The second component is continuous everywhere except at $t = -2$. It follows that the vector-valued function $\mathbf{r}$ is $\boxed{\text{continuous everywhere except at } t = -2.}$

(b) Compute $\mathbf{r}'(1)$.

Solution:
Differentiating $\mathbf{r}$, we have

$$\mathbf{r}'(t) = 3t^2\mathbf{i} - \frac{1}{(t+2)^2}\mathbf{j} - 2e^{2t}\mathbf{k}.$$

Hence

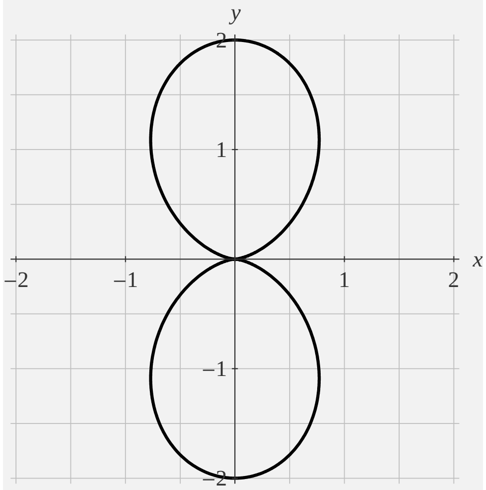
$$\boxed{\mathbf{r}'(1) = 3\mathbf{i} - \frac{1}{9}\mathbf{j} - 2e^2\mathbf{k}.}$$

(c) Evaluate $\displaystyle\int \mathbf{r}(t)\,\mathrm{d}t$.

Solution:
Integrating component by component yields

$$\begin{aligned}
\int \mathbf{r}(t)\,\mathrm{d}t &= \int \left(t^3\mathbf{i} + \frac{1}{t+2}\mathbf{j} - e^{2t}\mathbf{k}\right)\,\mathrm{d}t \\
&= \int t^3\,\mathrm{d}t\mathbf{i} + \int \frac{1}{t+2}\,\mathrm{d}t\mathbf{j} - \int e^{2t}\,\mathrm{d}t\mathbf{k} \\
&= \boxed{\left(\frac{1}{4}t^4 + C_1\right)\mathbf{i} + \left(\ln|t+2| + C_2\right)\mathbf{j} + \left(\frac{1}{2}e^{2t} + C_3\right)\mathbf{k}.}
\end{aligned}$$

5. A graph of the 2-leaf rose $r = 1 - \cos 2\theta$ is given.



(a) Compute the area of one leaf of the 2-leaf rose $r = 1 - \cos 2\theta$. JUSTIFY YOUR ANSWER.

Solution:
The limits of integration for the leaf in the upper half of the plane are 0 and π. Thus

$$A = \int_0^\pi \frac{1}{2}(1-\cos 2\theta)^2 \, \mathrm{d}\theta = \int_0^\pi \left(\frac{1}{2} - \cos 2\theta + \frac{1}{2}\cos^2 2\theta\right) \mathrm{d}\theta$$

$$= \int_0^\pi \left(\frac{3}{4} - \cos 2\theta + \frac{1}{4}\cos 4\theta\right) \mathrm{d}\theta = \left[\frac{3}{4}\theta - \frac{1}{2}\sin 2\theta + \frac{1}{16}\sin 4\theta\right]_0^\pi$$

$$= \boxed{\frac{3\pi}{4}.}$$

(b) Provide the equation for the tangent line to the 2-leaf rose $r = 1 - \cos 2\theta$ at the point with polar coordinates $(1, \frac{\pi}{6})$ (typo: this point should actually be $(\frac{1}{2}, \frac{\pi}{6})$).

Solution:
The parametric equations for the curve are

$$x = (1-\cos 2\theta)\cos\theta \text{ and } y = (1-\cos 2\theta)\sin\theta.$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin 2\theta \sin\theta + (1-\cos 2\theta)\cos\theta}{2\sin 2\theta\cos\theta - (1-\cos 2\theta)\sin\theta}.$$

At the point where $\theta = \frac{\pi}{6}$, the slope of the tangent line is

$$\frac{dy}{dx} = \frac{2\sin\frac{2\pi}{6}\sin\frac{\pi}{6} + (1-\cos\frac{2\pi}{6})\cos\frac{\pi}{6}}{2\sin\frac{2\pi}{6}\cos\frac{\pi}{6} - (1-\cos\frac{2\pi}{6})\sin\frac{\pi}{6}} = \frac{2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + (1-\frac{1}{2})\frac{\sqrt{3}}{2}}{2\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - (1-\frac{1}{2})\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}}{\frac{3}{2}-\frac{1}{4}} = \frac{3\sqrt{3}}{5}.$$

The cartesian coordinates for the point with polar coordinates $(\frac{1}{2}, \frac{\pi}{6})$ are $(\frac{1}{2}\cos\frac{\pi}{6}, \frac{1}{2}\sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{4}, \frac{1}{4})$. The equation for the tangent line is

$$\boxed{y - \frac{1}{4} = \frac{3\sqrt{3}}{5}\left(x - \frac{\sqrt{3}}{4}\right).}$$

This simplifies to $y = \frac{3\sqrt{3}}{5}x - \frac{1}{5}$.

6. The position of a spot on the circumference of a rolling tire is given by

$$\mathbf{r}(t) = (10t - \sin(10t))\mathbf{i} + (1 - \cos(10t))\mathbf{k}.$$

(a) Find the velocity vector for the spot.

(b) Find the acceleration vector for the spot.

(c) What is the maximum speed of the spot?