Note: Solutions to Sample Test 3

[#1] (a) Let $f(y) = -y^2(y-3)$. Equilibrium solutions are zero points of $f(y) = 0$. They are $y = 0, y = 3$. (b) For any initial point between the to equilibrium points $y = 0, y = 3$, $dy/dt > 0$. So the graph of the solution increases, approaching $y = 3$ asymptotically. (c) By Euler's method, $y_{n+1} = y_n + \Delta x f(y_n) = y_n + 0.5(-y_n^2(y_n - 3))$

n	t_n	y_n	$\Delta x f(y_n)$
0	0	1	$0.5(-1^2(1-3)) = 1$
1	0.5	$1+1=2$	$0.5(-2^2(2-3)) = 2$
2	1	$2+2=4$	–

Hence $y(1)$ 4.

[#2] (a) $\begin{cases} \dfrac{dy}{dt} = 5\text{g/L} \times 100\text{L/min} - \dfrac{y(t)}{1000}\text{g/L} \times 100\text{L/min} = 500 - \dfrac{y(t)}{10} \\ y(0) = 0 \end{cases}$

(b) By the method of separation of variables,

$$\begin{aligned}
\frac{dy}{dt} &= \frac{5000 - y(t)}{10} \\
\frac{dy}{5000 - y} &= \frac{1}{10}dt \\
\int \frac{dy}{5000 - y} &= \int \frac{1}{10}dt = \frac{t}{10} + C_1 \\
-\ln|5000 - y| &= \frac{t}{10} + C_1 \\
|5000 - y| &= e^{-\frac{t}{10} - C_1} = C_2 e^{-\frac{t}{10}} \\
5000 - y &= \pm C_2 e^{-\frac{t}{10}} = C_3 e^{-\frac{t}{10}} \\
y(t) &= 5000 - C_3 e^{-\frac{t}{10}} = 5000 + Ce^{-\frac{t}{10}}
\end{aligned}$$

Use the initial condition $y(0) = 0$, we have

$$0 = y(0) = 5000 + Ce^{-\frac{0}{10}} = 5000 + C \Longrightarrow C = -5000,$$

and

$$y(t) = 5000(1 - e^{-\frac{t}{10}}).$$

(c) The time t that happens satisfies

$$3000 = y(t) = 5000(1 - e^{-\frac{t}{10}}).$$

Solving for t to have: $3/5 = 1 - e^{-\frac{t}{10}}$, $e^{-\frac{t}{10}} = 1 - 3/5 = 2/5$, and $t = -10 \ln 2/5$ 9.2min.

[#3]

$$\begin{aligned}
&\frac{dy}{dx} = \frac{2xy}{x^2+1}\\
&\frac{dy}{y} = \frac{2x}{x^2+1}dx\\
&\int \frac{dy}{y} = \int \frac{2x}{x^2+1}dx\\
&\ln|y| = \ln(x^2+1) + C_1\\
&|y| = e^{\ln(x^2+1)+C_1} = C_2(x^2+1)\\
&y = \pm C_2(x^2+1) = C(x^2+1)
\end{aligned}$$

With $3 = y(0) = C(0^2+1) = C$ we have $y(x) = 3(x^2+1)$.

[#4] $A = (-1,1,2), B = (1,-1,0), C = (2,1,1)$. (a) $\vec{AB} = (2,-2,-2), \vec{BC} = (1,2,1)$. (b)$\vec{AB}\cdot\vec{BC} = 2(1) - 2(2) - 2(1) = -4$. $\vec{AB}\times\vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -2 \\ 1 & 2 & 2 \end{vmatrix} = 2\vec{i} - 4\vec{j} + 6\vec{k} = (2,-4,6)$. (d) No. Vectors parallel to $(-1,1,0)$ have the form $k(-1,1,0) = (-k,k,0)$. Since $\vec{AB}\cdot(-k,k,0) = 2(-k) - 2(k) - 2(0) = -4k \neq 0$ unless $k = 0$. (e) $(1/2)||\vec{AB}\times\vec{BC}|| = \sqrt{14}$. (f) $\cos\theta = \frac{\vec{AB}\cdot\vec{BC}}{||\vec{AB}||\cdot||\vec{BC}||} = -\frac{\sqrt{2}}{3}$ and $\theta = \arccos -\frac{\sqrt{2}}{3} = 2.06$ radians. (g) $(x-(-1), y-1, z-2)\times(\vec{AB}\times\vec{BC}) = 0$ which simplifies to $2(x+1) - 4(y-1) + 6(z-2) = 0 \Longrightarrow x - 2y + 3z = 3$. (h) $Proj_{\vec{BC}}\vec{AB} = \frac{\vec{AB}\cdot\vec{BC}}{||\vec{BC}||^2}\vec{BC} = -\frac{2}{3}\langle 1,2,1\rangle$.