Math 106 **Answers to Exam 3a** **Fall 2015**

. 101. Consider the curve given parametrically by $x(t) = \cos(2t), y(t) = (2t^3 - 1)^3$, for t from $-\pi$ to π.

(a) (6 points) Find all the points (x, y) where the graph has either a vertical or a horizontal tangent line.

(b) (4 points) Find the slope of the curve $\dfrac{dy}{dx}$ as a function of t.

Solution.

(a) In order to determine where the tangent to this curve is horizontal or vertical, we must examine the horizontal and vertical rates of change. We first find $\frac{dx}{dt} = -2\sin(2t)$. We then solve $0 = -2\sin(2t)$, which yields $t = -\frac{\pi}{2}$, $t = 0$, and $t = \frac{\pi}{2}$. These values correspond to the points $(-1, (\frac{-\pi^3}{4} - 1)^3)$, $(1, -1)$, and $(1, (\frac{\pi^3}{4} - 1)^3)$, respectively. Next, we find $\frac{dy}{dt} = 18t^2(2t^3 - 1)^2$. Solving $0 = 18t^2(2t^3 - 1)^2$ yields $t = 0$ and $t = \sqrt[3]{\frac{1}{2}}$. These values correspond to the points $(1, -1)$ and $(\cos(2\sqrt[3]{\frac{1}{2}}), 0)$, respectively. We now notice that when $t = 0$ (that is, at the point (1,-1)) both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. Determining the behavior of the curve at that point requires looking at a graph; doing so, we verify that the tangent at that point is indeed horizontal and not vertical. We then conclude that a vertical tangent at $(-1, (\frac{-\pi^3}{4} - 1)^3)$ and $(1, (\frac{\pi^3}{4} - 1)^3)$, and a horizontal tangent at $(1, -1)$ and $(\cos(2\sqrt[3]{\frac{1}{2}}), 0)$.

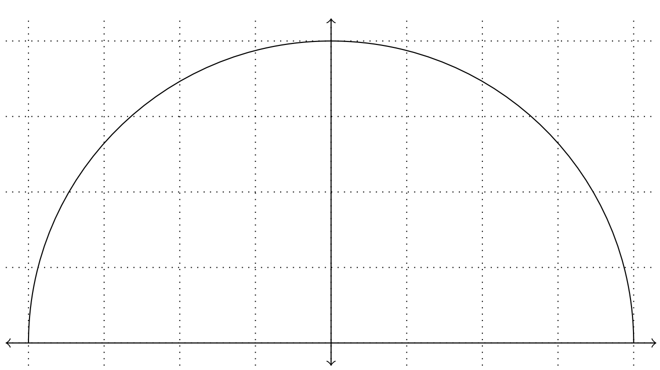
(b)
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{18t^2(2t^3 - 1)^2}{-2\sin(2t)} = \frac{-9t^2(2t^3 - 1)^2}{\sin(2t)}$$

grading:

(a) 2 points for correctly computing derivatives, 2 points for correctly finding zeros of those derivatives, 2 points for using the values of t found to generate (x, y) pairs. Students do not need to correctly classify the point $(1, -1)$ for full credit; students who acknowledge the issue at $t = 0$ and use a graph to classify that value should be awarded a bonus point.

(b) 2 points for attempting to take a ratio of dy/dt and dx/dt; 2 more points for doing so correctly.

. 102. Draw an appropriate sketch for $\displaystyle\int_{-4}^{4} \sqrt{16 - x^2}\, dx$ and then evaluate this integral.

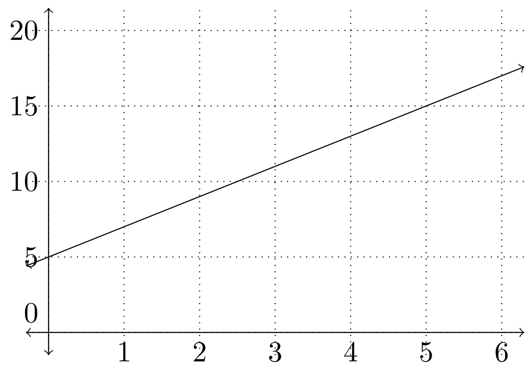


Solution. Graphing, we see that $y = \sqrt{16 - x^2}$ is the lower half of a circle centered at the origin of radius 4. (Recall, the area of a circle is given by πr^2.) Therefore
$$\int_{-4}^{4} \sqrt{16 - x^2}\, dx = \frac{4^2\pi}{2} = 32\pi.$$

grading: 4 points for a sketch of a half circle, 3 points for identifying the area of a circle, 3 points for evaluating.

12 3. At time t, in minutes, the velocity v of an object, in feet per minute is given by: $v(t) = 5 + 2t$. **Draw** the graph of $v(t)$ and **Draw** a right-hand sum to estimate the distance traveled from $t = 0$ to $t = 6$ using $n = 3$ subdivisions. **Write** down an expression for the sum. You do **not** need to add up the sum. **Explain** in a sentence if your estimate is an underestimate or an overestimate.

Solution.



Note that dividing the interval [0, 6] into 3 subdivisions yields intervals of length 2. We have $v(0) = 5, v(2) = 9, v(4) = 13, v(6) = 17$. We construct a left sum to approximate the distance covered for $0 \leq t \leq 6$ by approximating the area under the graph of v, which is:

$$[v(0) + v(2) + v(4)] \cdot 2 = (5 + 9 + 13) \cdot 2 = 54.$$

The right sum is given by

$$[v(2) + v(4) + v(6)] \cdot 2 = (9 + 13 + 17) \cdot 2 = 78.$$

The function $v(t)$ is increasing so the left sum will underestimate the distance while the right sum will overestimate the distance. Indeed, the **exact** distance covered is $\int_0^6 (5 + 2t)dt = (5t + t^2)|_0^6 = 66$.

Grading guidelines: 2 points for the graph of v; 2 points for shading the region; 5 points for writing the sum; 3 points for explaining if it is an underestimate or an overestimate.

8 4. Compute the following derivative: $\dfrac{d}{dy} \displaystyle\int_3^y \ln(1 + x^2) + \cos(3 - x)dx$, briefly justifying your answer.

Solution. State that they are using the fundamental theorem of calculus, part 2 [2 points], which says that for a continuous function on an interval $F(x)$, such that $F'(x) = f(x)$, we have that

$$F(x) = \int_a^x f(t)dt.$$

Then we have that $F(y) = \int_3^y \ln(1 + x^2) + \cos(3 - x)dx$. And we are suppose to take the derivative with respect to y, so $\dfrac{d}{dy}F(y)$ is exactly the function we were taking the integral of, so $\dfrac{d}{dy} \displaystyle\int_3^y \ln(1 + x^2) + \cos(3 - x)dx = \ln(1 + y^2) + \cos(3 - y)$. [Which is our other 6 points].

· 105. Evaluate exactly, clearly stating any properties or rules that you use.

(a) $\lim\limits_{t\to 0} \dfrac{e^t - 1 - t}{t^2}$.

(b) $\lim\limits_{t\to\infty} x\sin(1/x)$.

Solution.

(a) We first check that we have an indeterminate form in quotient form. Indeed, by letting $t \to 0$ we obtain $\dfrac{0}{0}$. We apply L'Hôpital's rule and obtain

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t\to 0} \frac{e^t - 1}{2t}.$$

We have again $\dfrac{0}{0}$, so we apply L'Hôpital's rule again and obtain

$$\lim_{t\to 0} \frac{e^t - 1}{2t} = \lim_{t\to 0} \frac{e^t}{2} = \frac{1}{2}.$$

grading guidelines for giving partial credit: 1 point for checking that we have an indeterminate form, 2 points for applying L'Hôpital each time. If the solution is correct give full credit but subtract 1 point for missing explanations.

(b) We have an indeterminate form $\infty \cdot 0$, we transform it in a quotient form by writing $\lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x}$. We now have the indeterminate form $\dfrac{0}{0}$ for which we can apply L'Hôpital in this form, or after making a change of variables $y = \frac{1}{x}$. With the change of variables, note that we have $y \to 0$ when $x \to \infty$ so we compute

$$\lim_{y\to 0} \frac{\sin y}{y} = \lim_{y\to 0} \frac{\cos y}{1} = 1.$$

Grading guidelines for giving partial credit: 1 point for checking that we have the indeterminate form $0 \cdot \infty$, 2 points for changing it a fraction, 2 points for applying L'Hôpital. If the solution is correct, give full credit but subtract 1 point for missing explanations.

grading:

· 106. A seed is dropped from the Sower statue on top of the Nebraska State Capitol, which is 400 feet above the ground. How long will it take the seed to reach the ground? Recall that $g = -32$ ft/s^2. (Ignore air resistance.)

Solution. Finding the velocity function: 4 points

- (1 pt) Using the acceleration function: $a = \dfrac{dv}{dt} =?32$
- (1 pt) Integrating both sides with respect to t: $\int \dfrac{dv}{dt}\,dt = \int ?32t\,dt$, and so $v =?32t + C$
- (1 pt) Finding C: $v(0) = 0$ so $0 =?32?0 + C$ so $C = 0$

- (1 pt) The velocity equation: $v =?32t$

Finding the position function: 4 points

- (1 pt) Using the velocity function: $v = \dfrac{ds}{dt} =?32t$
- (1 pt) Integrating both sides with respect to t:

$$\int \frac{ds}{dt}\,dt = \int ?32t\,dt, quad s = -16t^2 + C$$

- (1 pt) Finding C: $s(0) = 400$ so $400 =?16?0 + C$ so $C = 400$.
- (1 pt) The position function: $s =?16t^2 + 400$.

Finding time to hit ground: 2 points

(a) (1 pt) Setting $s = 0$ and solving: $?16t^2 + 400 = 0$ so $t^2 = 400/16$. Thus, $t^2 = 25$ and so $t = 5$ (

(b) (1 pt) Correct answer: t = 5

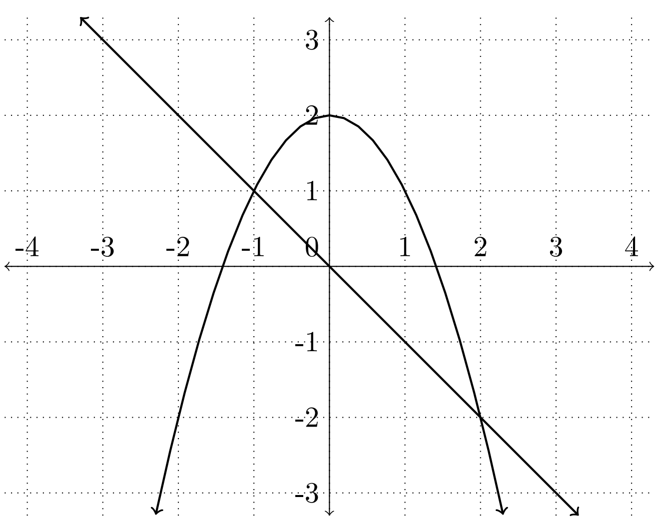
Additional notes:

(a) (5 pts) If a student just uses the formula $s(t) =?\frac{1}{2}gt^2 + v_0t + s_0$ – and plugs in correctly to get $s =?16t^2 + 400$.

(b) (8 pts) If a student does as above and also finds the correct answer.

· 127. Sketch the curves $y = 2 - x^2$ and $y = -x$ and find the exact area between them.

Solution.



4 points on the graph: 1 for parabola, 1 for line, 1 for intersection at $(-1, 1)$, 1 for $(2, -2)$.

3 points for setting up the integral (1 for "-1", 1 for "2", 1 for integrand) but -1 for omitting dx.

$$\begin{aligned}\int_{-1}^{2}(2-x^2)-(-x)\,dx &= \int_{-1}^{2}-x^2+x+2\,dx\\ &= -\frac{x^3}{3}+\frac{x^2}{2}+2x\Big|_{-1}^{2}\\ &= \left(-\frac{2^3}{3}+\frac{2^2}{2}+4\right)-\left(-\frac{-1}{3}+\frac{1}{2}-2\right)\\ &= \frac{9}{2}=4.5\end{aligned}$$

3 points for each antiderivative term in the first line and no penalty for omitting the vertical line.

2 points for evaluation and final answer (1 for work; 1 for answer)

10 8. Find the average value of $y = 1 + e^x$ over the interval $[0, 2]$.

Solution. The average value formula, for a function $f(x)$ over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

[4 points for starting with the right formula] So, we plug these in and get the average value to be

$$\frac{1}{2-0}\int_0^2 1+e^x dx.$$

$$=\frac{1}{2}\left[x+e^x\right]_0^2$$

[4 points for correct antiderivative]

$$=\frac{1}{2}\left[2+e^2\right]-\frac{1}{2}\left[0+e^0\right]$$

$$=\frac{1}{2}\left[2+e^2\right]-\frac{1}{2}$$

$$=\frac{1+e^2}{2}$$

[2 points for correct simplification]

$$\approx 4.195$$

[Either exact or approximate solutions is acceptable]

6 9. The function $y = f(x)$ is even and $\displaystyle\int_{-3}^{3}(f(x)-1)\,dx = 14$. Clearly explaining your work, find $\displaystyle\int_0^3 f(x)\,dx$.

Solution. Observe that since the function is even,

$$\int_{-3}^{3} f(x)\,dx = 2\int_0^3 f(x)\,dx$$

Observe, by linearity of integration,

$$\int_{-3}^{3} (f(x) - 1)\, dx = \int_{-3}^{3} f(x)\, dx - \int_{-3}^{3} 1\, dx = \int_{-3}^{3} f(x)\, dx - 6.$$

Putting this together, we have that

$$14 = 2\int_{0}^{3} f(x)\, dx - 6$$

Thus, $\int_0^3 f(x)\, dx = 20/2 = 10$.

grading: 2 points for each of the observations and 2 points for setting up and solving the final equation. Be generous with incomplete work.

12 10. Compute the following exactly, showing the details of your work. Do not use a calculator

(a) $\displaystyle\int_1^3 \pi\sqrt{x} - \sin x + \frac{e}{x}\, dx$

(b) $\displaystyle\int 2\cos x + x^{-7/2}\, dx.$

Solution.

(a)

$$\begin{aligned}\int_1^3 \pi\sqrt{x} - \sin x + \frac{e}{x}\, dx &= \pi\frac{x^{3/2}}{3/2} + \cos x + e\frac{x^{-2}}{-2}\Bigg|_1^3 \\ &= \pi\frac{2}{3}x^{3/2} + \cos x - \frac{e}{2x^2}\Bigg|_1^3 \\ &= \left(\pi\frac{2}{3}2^{3/2} + \cos 2 - \frac{e}{8}\right) - \left(\pi\frac{2}{3} + \cos 1 - \frac{e}{2}\right) \\ &= \frac{\pi(2^{5/2} - 1)}{3} + \cos 2 - \cos 1 + \frac{3e}{8}\end{aligned}$$

grading: 6 points total: 2 points for first anti-derivative term; 1 each for second and third terms; 2 for plugging in endpoints (to whatever anti-derivatives are used); nothing for simplifying

(b)

$$\begin{aligned}\int 2\cos x + x^{-7/2}\, dx &= 2\sin x + \frac{x^{-5/2}}{-5/2} + C \\ &= 2\sin x - \frac{2}{5x^{5/2}} + C\end{aligned}$$

grading: 6 points total; 2 points for each anti-derivative term; 2 points for "+C"