Solutions to Exam 2

1. **(version A.)** Find $\dfrac{dy}{dx}$ given that $e^{3y} + \cos(4x^2) = 7x^2y^3 - 19$.

   Solution. By differentiation we obtain

$$3e^{3y}\frac{dy}{dx} - 8x\sin(4x^2) = 14xy^3 + 21x^2y^2\frac{dy}{dx}$$

   from which we solve for $\frac{dy}{dx}$ to find

$$\frac{dy}{dx} = \frac{8x\sin(4x^2) + 14xy^3}{3e^{3y} - 21x^2y^2}.$$

2. **(version B.)** Given the function

$$y = \frac{ax}{b^2 + x^2}$$

   with a and b positive, find a and b so that the point $(-5,-2)$ is a global minimum and the point $(5,2)$ is a global maximum.

   Solution. For the function $y(x) = \dfrac{ax}{b^2 + x^2}$ we have that

$$y'(x) = \frac{a(b^2 + x^2) - 2ax^2}{(b^2 + x^2)^2}.$$

   Thus the critical points would satisfy the equation $ab^2 - ax^2 = 0$ which after factoring is $a(b^2 - x^2) = 0$; so solutions are $x = \pm b$. We are given that the points $(-5,-2), (5,2)$ are critical points, hence we have that $b = 5$ (since $b > 0$). By using the fact that $y(5) = 2$ we have that $2 = \dfrac{5a}{50}$ so $a = 20$.

3. **(version A.)** A rectangle with horizontal and vertical sides has two vertices on the x-axis and two vertices on the curve

$$y = e^{-3x^2}.$$

   Find the vertices of the rectangle with maximum area.

   Solution. This is a variation on Q12 from 4.3.

   Observe that rectangle has area $A = 2xe^{-3x^2}$ and $A' = 2e^{-3x^2} - 12x^2e^{-3x^2}$. Thus, $A' = 0$ means $1 - 6x^2 = 0$ and so $x = 1/\sqrt{6}$. Use the first derivative test to show since

   - the derivative is positive (so A is increasing) for values of $x < 1/\sqrt{6}$
   - $A' < 0$ (so A is decreasing) for $x > 1/\sqrt{6}$

   to deduce that A reaches a maximum at $x = 1/\sqrt{6}$. The vertices of the rectangle are $(1/\sqrt{6}, e^{-1/2}), (-1/\sqrt{6}, e^{-1/}$

4. **(version A.)** Use the racetrack principle and $\sin 0 = 0$ to show that $\sin x \le x$ for all $x \ge 0$.

   Solution. Let $f(x) = x$ and $g(x) = \sin x$. Since

   - both functions have the same starting point (due to the fact that $f(0) = 0 = g(0)$)
   - the slope of f is always above the slope of g (indeed: $f'(x) = 1 \ge \cos x = g'(x)$ for all x)

by the racetrack principle we have that the values of f must always be above the values of g for $x \geq 0$, hence $\sin x \leq x$ for all $x \geq 0$.

5. **(version A.)** Let $f(x) = 2x^3 + 3x^2 - 36x$.

   (a) Find all critical points of $f(x)$.
   We have that $f'(x) = 6x^2 + 6x - 36$ so the critical points will be solutions of the equation $6x^2 + 6x - 36 = 0$; these roots are $x = 2$, $x = -3$.

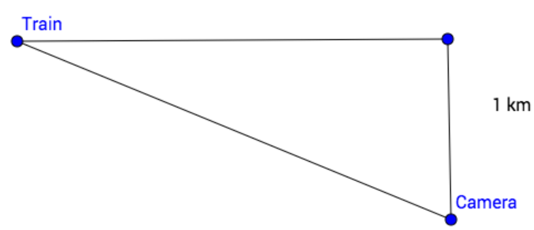
   (b) Classify the critical points of $f(x)$ as local maxima, local minima, or neither. Justify your answers.
   We have the following table summarizing the first derivative test for f:

x		-3		-2	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	incr.		decr.		incr.

   Thus -3 is a local maximum and -2 is a local minimum.

   (c) Find all inflection points of $f(x)$. As always, justify your answers.
   We compute that $f''(x) = 12x+6$ which means that the inflection points would satisfy $12x+6 = 0$, i.e. $x = -\dfrac{1}{2}$.

6. (20 points) **(version A.)** A train is traveling at 2 km/min east along a long straight track. A camera is one kilometer south of the track and is east of the train (i.e., the train is getting closer to the camera). How fast is the distance from the train to the camera decreasing when the train is exactly 3 km from the camera?



This is a variation on Q44 from Section 4.6.

Solution. Let $x(t)$ be the distance between the train and the point on the track that is directly north of the camera (call it point B), and $z(t)$ be the distance from the train to the camera. We want to find dz when $z = 2$. By the Pythagorean theorem, we have that

$$x^2(t) + 1 = z^2(t).$$

Use implicit differentiation to get

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt},$$

so

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}.$$

Since the train is traveling at 3 km/min east then the distance between the train and point B is decreasing by 3 km/min. So $\frac{dx}{dt} = -3$ km/min. We are given that $z = 2$, so we see that

$$x = \sqrt{2^2 - 1} = \sqrt{3}.$$

Then

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = -\frac{3\sqrt{3}}{2}.$$

7. (12 points) **(version A.)**

   (a) (5 pts) Find the linear approximation, $L(x)$ of $f(x) = x^3 - bx^2 - x + 4$ near $x = 0$, where b is a positive constant.

   (b) (2 pts) Use your linear approximation to estimate $f(x)$ at $x = 1$. That is, find $L(1)$.

   (c) (5 pts) Is $L(1)$ above or below the actual value of $f(x)$ at $x = 1$?

   Solution. As $f'(0) = -1$, we have that $L(x) = 4 - x$, so $L(1) = 3$. We have that $L(1) < f(1)$ as $f''(0) = -2b < 0$ and so the graph is concave down (i.e., graph of function is below the graph of the tangent line).

8. (15 points) **(version B)**

   (a) (8 pts) The population of Lincoln, in thousands, is given by a function $P = f(t)$ where t is years since 2000. Explain the meaning of the following statements, in complete sentences.

      (i) (3 pts) $f(5) = 242$.
      The population of Lincoln was 242,000 in 2005.

      (ii) (3 pts) $f'(13) = 3.2$
      In 2013 the rate of change for the population was 3,200 per year, i.e. the population increased by about 3,200 in 2013.

      (iii) (2 pt) $f^{-1}(260) = 13$
      It took about 13 years (since 2000) for the population to reach 260,000. Or, the year in which the population reached 260,000 was 2013.

   (b) (7 points) Two points for each of the critical points ($x = -2$, $x = 1.5$); 1 point for correctly identifying the monotonicity of f (increasing or decreasing) on each interval.