Math 106 **Answers to Exam 1a** **Fall 2015**

12 1. Find the derivative of the following functions. **Do not simplify your answers.**

(a) $f(x) = \dfrac{e^x - \cos x}{x^2 + 2}$

(b) $g(z) = \left[3\sin(z^2) + e^z\right]^5$

Solution. Using the quotient rule on $f(x)$ and the chain rule on $g(z)$, we have

$$f'(x) = \frac{(e^x + \sin x)(x^2+2) - (e^x - \cos x)2x}{(x^2+2)^2}$$
$$g'(z) = 5\left[3\sin(z^2) + e^z\right]^4\left(3\cos(z^2)2z + e^z\right)$$

for (a), 2 points for setting up the chain rule form, 2 points for the derivative of the numerator and 2 for the derivative of the denominator; for (b), 3 points for setting up the first chain rule correctly, 2 points for the derivative of $3\sin(z^2)$ *and 1 point for the derivative of* e^z*.*

10 2. By July 1st of 1915 the United States population was 100 million. By July 1st of 1968 the United States population was 201 million.

(a) (5 pts) Assuming the population of the United States is exponential and using 1915 as year zero, find a formula for this growth of the form: $P(t) = P_0 e^{kt}$.

(b) (5 pts) By July 1st of what year will the US population reach 300 million, according to your formula?

Solution.

(a)

$$\begin{aligned} P(t) &= 100e^{kt} \\ 201 &= 100e^{53k} \\ 2.01 &= e^{53k} \\ k &= \ln 2.01/53 \end{aligned}$$

$$P(t) = 100e^{0.0132k}$$

(b)

$$\begin{aligned} 300 &= 100e^{0.0132k} \text{ or } 100e^{\ln(2.01)/53k} \\ 3 &= e^{0.0132k} \\ k &= \ln(3)/0.0132 = 83.403 \end{aligned}$$

So the US population will reach 300 million by July 1, 1998.

Part (a): 2 points for setup, 2 points for correct use of logs, 1 point for correct answer. Part (b): 2 points for setup, 2 points for use of logs, 1 point for correct answer.

10 3. Use the figure below to estimate the following limits. Write **DNE** if the limit does not exist.

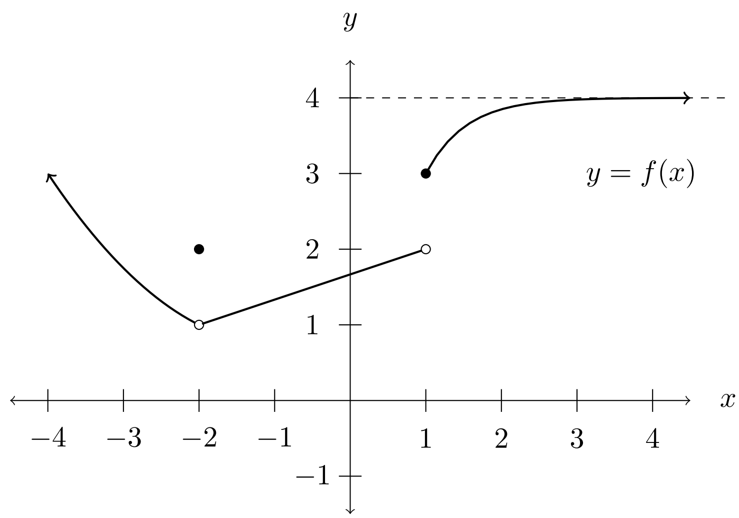
(a) $\lim_{x \to -2} f(x)$

(b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$

(d) $\lim_{x \to 1} f(x)$

(e) $\lim_{x \to \infty} f(x)$



Solution.

(a) 1
(b) 2
(c) 3
(d) DNE
(e) 4

2 points each

12 4. Let $R = f(a)$ be the revenue, in dollars, that a company makes from spending a dollars on advertising. Using complete sentences, answer the following questions

(a) What does the sign of $f'(a)$ tell you about the company's advertising?
(b) What does it mean to write $f'(100,000) = 1/2$?
(c) If $f'(100,000) = 1/2$, then would you recommend changing the advertising budget? Why?

Solution.

(a) If $f'(a)$ is positive, then the company's revenue increases when they spend a dollars on advertising. If $f'(a)$ is negative, then the company's revenue decreases when they spend a dollars on advertising.
(b) If $f'(100,000) = 1/2$, then each dollar above \$100,000 spent on advertising will bring in \$0.50 worth of sales.

(c) If $f'(100,000) = 1/2$, then the increase in revenue is less than the additional expense, hence too much is being spent on advertising. So we would recommend decreasing the advertising budget.

3 points per part. Part (c) needs to include justification.

10 5. Find the value of k that makes the following function continuous.

$$f(x) = \begin{cases} kx - 1, & \text{for } x < 2, \\ (3-x)^2 + 1, & \text{for } x \geq 2. \end{cases}$$

Solution. The function has two branches, so we will have to compute the side limits.

$$\lim_{x \to 2^-} f(x) = 2k - 1 \text{(2 points)}$$

while

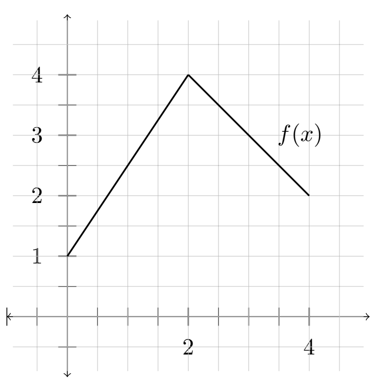
$$\lim_{x \to 2^+} f(x) = (3-2)^2 + 1 = 2. \text{(2 points)}$$

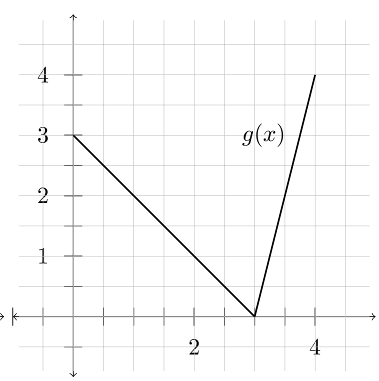
In order for the function to be continuous the two values must match and they must match the value of the function; in other words, we must have

$$2k - 1 = 2 = f(2).$$

The above equality takes place for $k = 3/2$, hence the function is continuous only for $k = 3/2$. (1 point)

12 6. Use the graphs of $f(x)$ and $g(x)$ below to compute the derivatives below. As always, clearly show your work.





(a) If $h(x) = f(x)g(x)$, compute $h'(1)$.

(b) If $k(x) = g(x)/f(x)$, compute $k'(2.5)$.

(c) If $l(x) = f(g(x))$, compute $l'(2)$.

Solution.

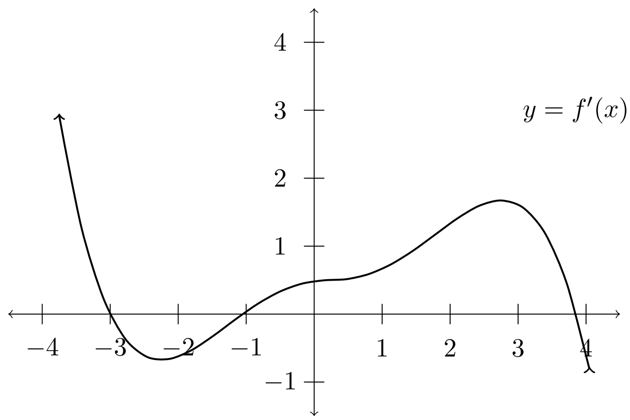
(a) $h'(x) = f'(x)g(x) + f(x)g'(x)$ so $h'(1) = f'(1)g(1) + f(1)g'(1) = (1.5)(2) + (2.5)(-1) = 0.5$

(b) $k'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2}$, so $k'(2.5) = \frac{(3.5)(-1) - (0.5)(-1)}{(3.5)^2}$

(c) $l'(x) = f'(g(x))g'(x)$, so $l'(x) = f'(1)(-1) = -1.5$

Each part should be 4 points; 2 points for evidence of product/quotient/chain rule, 2 points for correct values/arithmetic.

147. The following graph is of **the derivative** of a function $y = f(x)$. Using the graph, answer the following questions **and justify each answer**.



(a) In the interval $[-3, 3]$, for which x value does $y = f(x)$ have the largest **slope**?

(b) In the interval $[-3, 3]$, for which x value does $y = f(x)$ have its smallest **value**?

(c) Give any intervals where $y = f(x)$ is increasing.

(d) Give any intervals where $y = f''(x)$ is positive.

(e) Give any intervals where $y = f(x)$ is concave up.

Solution. For part (a), $y = f(x)$ has the largest slope when $y = f('x)$ has the largest value. On the interval $[-3, 3]$, the largest value of the derivative is at about 2.8 or so.

For (b), $y = f(x)$ has its smallest value when $y = f'(x)$ stops decreasing, i.e., at the end of an interval when the derivative is negative. Thus, the smallest value of the function occurs when $x = -1$.

For (c), the function $y = f(x)$ is increasing when f' is positive. So the intervals are $(-\infty, -3)$ and $(-1, 3.8)$.

For (d), the second derivative $y = f''(x)$ is positive when $y = f'(x)$ is increasing, so the answer is the interval $(-2, 2.8)$.

For (e), the function is concave up when the second derivative is positive, so this question is asking for the same thing as part (d) and has the same answer, $(-2, 2.8)$.

3 points for each of the first four parts and 2 points for part (e)

108. Find the equation of tangent line to the graph of the following function at $x = -1$.

$$f(x) = x^5 + 3x^2 - 4$$

Solution. First, we evaluate the function at $x = -1$, namely $f(-1) = (-1)^5 + 3(-1)^2 - 4 = -1 + 3 - 4 = -2$. Thus the tangent line goes through the point $(-1, -2)$. Next, we find the derivative function $f'(x) = 5x^4 + 6x$ and then evaluate it at $x = -1$, giving $f'(-1) = 5(-1)^4 + 6(-1) = -1$. Thus the slope of the tangent line is -1. Putting this together, the equation of the tangent line is

$$y + 2 = (-1)(x + 1).$$

If you want, you can simplify this to obtain $y = -x - 1$.

109. Using **the definition of the derivative in terms of a limit**, find the derivative of $f(x) = x^2 - 1$.

Solution. The definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Next, we simplify the numerator

$$f(x+h) - f(x) = ((x+h)^2 - 1) - (x^2 - 1) = x^2 + 2xh + h^2 - 1 - x^2 + 1 = 2xh + h^2 = h(2x + h)$$

and so

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h = 2x. \end{aligned}$$

Note that you can check this answer by either using the derivative rules and computing $f'(x) = 2x$.

2 points for the definition of derivative, 2 points for substituting in x+h in the function, 1 point for the rest of simplifying the numerator, 3 points for simplifying the ratio correctly in the definition of derivative and 2 points for the final answer.