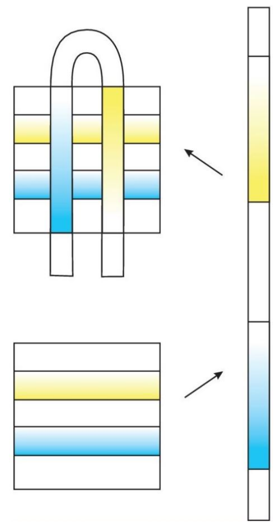
Horseshoe Map Theorem



Let $Q = [0,1] \times [0,1]$, and $f : Q \to \mathbb{R}^2$ be a diffeomorphism as shown in the graph. We are interested in the set of points

$$\Delta := \{(x,y) \in Q : f^k(x,y) \in Q \text{ for all } k \in \mathbb{Z}\}.$$

that stay in Q for all forward and backward iterates under f. As it is shown, the domain of f whose image is in Q consists of two horizontal strips

$$D = f^{-1}(Q) \cap Q = H_1 \cup H_2.$$

And the corresponding range consists of two vertical strips

$$R = f(Q) \cap Q = V_1 \cup V_2.$$

We have

$$V_i = f(H_i) \text{ and symmetrically } H_i = f^{-1}(V_i),\ i = 1, 2.$$

As a visual device, you may use $V_1 = V_L$ for the left v-strip and $V_2 = V_R$ for the right v-strip, and H_L is the bottom h-strip and H_R is the top h-strip. As a consequence

$$\Delta \subset (H_1 \cup H_2) \cap (V_1 \cup V_2)$$

because the iterates of points from Δ must stay in the restricted domain and range of f to Q.

Definion 1. *If $\Delta \neq \varnothing$, then each $p \in \Delta$,*

$$\phi(p) = (\cdots s_{-m} \cdots s_{-1}.s_0 s_1 \cdots s_n \cdots)$$

is called the itinerary of p if

$$f^k(p) \in H_{s_k} \text{ for all } k \in \mathbb{Z}.$$

Definion 2. *(1) For each $p \in D$, define*

$$\phi^+(p) = s_0 s_1 \cdots s_n \cdots$$

if $f^k(p) \in H_{s_k}$ for all $k \geq 0$, and call it the forward itinerary of p.

(2) For each $p \in R$, define

$$\phi^-(p) = \cdots s_{-m} \cdots s_{-1} s_0$$

if $f^k(p) \in V_{s_k}$ for all $k \leq 0$, and call it the backward itinerary of p.

(3) For each forward finite itinerary $s_0 \cdots s_n$, define

$$H_{s_0 \cdots s_n} = \{p \in Q : f^k(p) \in H_{s_k},\ k = 0, 1, \dots, n\}.$$

(4) For each backward finite itinerary $s_{-m} \cdots s_0$, define

$$V_{s_{-m} \cdots s_0} = \{p \in Q : f^k(p) \in V_{s_k},\ k = 0, -1, \dots, -m\}.$$

The following the result can be verified directly.

Proposition 1. *(a) For every finite forward itinerary, $H_{s_0 \cdots s_n}$ is non-empty and*

$$\begin{aligned} H_{s_0 \cdots s_n} &= H_{s_0 \cdots s_{n-1}} \cap f^{-n}(H_{s_n}) \\ H_{s_0 \cdots s_n} &= H_{s_0} \cap f^{-1}(H_{s_1 \cdots s_n}) \\ H_{s_0 \cdots s_{n+1}} &\subset H_{s_0 \cdots s_n} \end{aligned}$$

(b) For every finite backward itinerary, $V_{s_{-m} \cdots s_{-1} s_0}$ is non-empty and

$$\begin{aligned} V_{s_{-m} \cdots s_{-1} s_0} &= V_{s_{-m+1} \cdots s_{-1} s_0} \cap f^m(V_{s_{-m}}) \\ V_{s_{-m} \cdots s_{-1} s_0} &= V_{s_0} \cap f(V_{s_{-m} \cdots s_{-1}}) \\ V_{s_{-(m+1)} \cdots s_{-1} s_0} &\subset V_{s_{-m} \cdots s_{-1} s_0} \end{aligned}$$

Proposition 2. *Assume for each infinite forward itinerary sequence,*

$$H_{s_0 \cdots s_n \cdots} = \bigcap_{k=0}^{\infty} H_{s_0 \cdots s_k}$$

is a unique horizontal curve (i.e., the graph of a function of $0 \le x \le 1$). Assume for each infinite backward itinerary sequence,

$$V_{\cdots s_{-m} \cdots s_{-1}} = \bigcap_{k=1}^{\infty} V_{\cdots s_{-k} \cdots s_{-1}}$$

is a unique vertical curve (i.e., the graph of a function of $0 \le y \le 1$). Then the intersection

$$V_{\cdots s_{-m} \cdots s_{-1}} \cap H_{s_0 \cdots s_n \cdots} = \{p\}$$

is a unique point whose itinerary is exactly

$$\phi(p) = s = (\cdots s_{-m} \cdots s_{-1} . s_0 s_1 \cdots s_n \cdots).$$

Proof. By definition, $f^k(p) \in H_{s_k}$ for $k \ge 0$ because $p \in H_{s_0 \cdots s_n \cdots}$ for all $n \ge 0$. Also, by the definition for backward itinerary, $p \in V_{s_{-1}},\ f^{-1}(p) \in V_{s_{-2}}$, and $f^{-k}(p) \in V_{s_{-(k+1)}}$ for all $-k \le 0$. This is equivalent to $f^{-1}(p) \in f^{-1}(V_{s_{-1}}) = H_{s_{-1}},\ f^{-2}(p) \in f^{-1}(V_{s_{-2}}) = H_{s_{-2}}$, and in general $f^{-k}(p) = f^{-1}(f^{-(k-1)}(p)) \in f^{-1}(V_{s_{-(k)}}) = H_{s_{-(k)}}$. Hence $\phi(p) = s$ by definition. □

The question now becomes under what conditions do the assumptions of the proposition above hold and does the itinerary mapping ϕ defines a topological conjugacy to the shift map on the symbolic space of doubly infinite sequences.

Here below the shift dynamics of symbolic system $\sigma : \Sigma_2 \to \Sigma_2$ is defined as follows:

$$\Sigma_2 = \{0,1\}^{\mathbb{Z}} = \{s = (\dots s_{-1}.s_0 s_1 \dots) : s_k \in \{0,1\},\ k \in \mathbb{Z}\}$$

and for every $s \in \Sigma_2$,

$$\sigma(s) = (\dots s'_{-1}.s'_0 s'_1 \dots),\ s'_k = s_{k+1},\ k \in \mathbb{Z}.$$

Also, for every $a > 1$ and all $s, s' \in \Sigma_2$,

$$d_a(s,s') = \sum_{i=-\infty}^{\infty} \frac{\delta(s_i, s'_i)}{a^{|i|+1}},$$

defines a complete metric and they are topologically equivalent for all $a > 0$, meaning a sequence $s^{(n)}$ of Σ_2 is convergent with metric d_a iff it is convergent with metric d_b as long as $a > 1, b > 1$. Also, s' is in a small neighborhood of s iff s' and s have the same symbols $s'_k = s_k$ for $-M \le k \le N$ for some sufficiently large $M, N > 1$. Here $\delta(s,t)$ defines the discrete metric on the symbol space $\{0,1\}$.

Theorem 1. *Let $Q = [0,1] \times [0,1]$, and $f : Q \to \mathbb{R}^2$ be a diffeomorphism. Let*

$$\Delta := \{(x,y) \in Q : f^k(x,y) \in Q \text{ for all } k \in \mathbb{Z}\}.$$

Assume

(a) The pre-image $f^{-1}(Q) \cap Q$ consists of two connected components, called H_1 and H_2, whose images are denoted by $V_1 = f(H_1), V_2 = f(H_2)$. H_i and V_j are assumed to be the horizontal strips and vertical strips, respectively, in the sense that for each $i = 1,2$, and every $\bar{y} \in [0,1]$, the y-cross-section $V_i \cap \{y = \bar{y}\}$ of V_i is nonempty and its pre-image $f^{-1}(V_i \cap \{y = \bar{y}\})$ in H_i is a graph over the x-interval $[0,1]$.

(b) f is contractive in the x-direction and expansive in the y-direction in the sense that

$$||D_j f_{1,i}|| < \frac{1}{2} \qquad \text{and} \qquad ||D_j g_{2,i}|| < \frac{1}{2},$$

where $f\big|_{H_i} = (f_{1,i}, f_{2,i})$, $f^{-1}\big|_{V_i} = (g_{1,i}, g_{2,i})$ and $V_i = f(H_i)$ for $i, j = 1, 2$.

Then the dynamical system $\{f, \Delta\}$ is topologically conjugate to the shift dynamics $\{\sigma, \Sigma_2\}$. That is, there is a homeomorphism $\phi : \Delta \to \Sigma_2$ so that

$$\phi \circ f = \sigma \circ \phi$$

Proof. We claim first that under the assumption there exists a constant $0 < \lambda < 1$ and four differentiable functions $h_{1,i}, h_{2,i}, i = 1, 2$, mapping Q into itself such that

$$||D_j h_{1,i}|| + ||D_j h_{2,i}|| \leq \lambda < 1 \tag{1}$$

for all $i,\ j = 1,\ 2$; and, more importantly, for $(\bar{x}, \bar{y}) = (f_{1,i}, f_{2,i})(x, y)$ with $(x, y) \in H_i$ it is equivalent to the following cross representations

$$\begin{array}{l} y = h_{2,i}(x, \bar{y}) \\ \bar{x} = h_{1,i}(x, \bar{y}) \end{array}$$

Assume this claim and consider first an orbit γ of f through $(x_0, y_0) \in \Delta$. By definition, the itinerary $s = (\cdots s_{-1}.s_0 s_1 \cdots)$ for (x_0, y_0) is uniquely determined by the rule $(x_k, y_k) \stackrel{\text{def}}{=} f^k(x_0, y_0) \in H_{s_k}$ for all $k \in \mathbb{Z}$. Using the cross representations above with $(x, y) = (x_k, y_k)$ and $(\bar{x}, \bar{y}) = (x_{k+1}, y_{k+1})$, i.e., $y_k = h_{2,s_k}(x_k, y_{k+1}), x_{k+1} = h_{1,s_k}(x_k, y_{k+1})$, and appending the resulting expressions for all $k \in \mathbb{Z}$ yield

$$\begin{array}{rcl} & \vdots & \\ x_{-1} & = & h_{1,s_{-2}}(x_{-2}, y_{-1}) \\ y_{-1} & = & h_{2,s_{-1}}(x_{-1}, y_0) \\ x_0 & = & h_{1,s_{-1}}(x_{-1}, y_0) \\ y_0 & \stackrel{*}{=} & h_{2,s_0}(x_0, y_1) \\ x_1 & = & h_{1,s_0}(x_0, y_1) \\ y_1 & = & h_{2,s_1}(x_1, y_2) \\ & \vdots & \end{array}$$

Treat the right hand side as an operator, say $\Phi(\cdot, s)$, mapping the doubly product space $Q^{\mathbb{Z}}$ into itself and the asterisk above the equal sign the center of the doubly infinite system. Then $(x_0, y_0) \in \Delta$ implies $\zeta = (\cdots \zeta_{-1}.\zeta_0 \zeta_1 \cdots) \in Q^{\mathbb{Z}}$ with $\zeta_k = (y_k, x_{k+1})$ must be a fixed point of $\Phi(\cdot, s)$, and conversely, if ζ is a fixed point for $\Phi(\cdot, s)$, then there must be an orbit $\{(x_k, y_k) = f^k(x_0, y_0) \big| k \in \mathbb{Z}\}$ so that $\zeta_k = (y_k, x_{k+1})$ is true for all $k \in \mathbb{Z}$ because of the cross representations. Moreover, if the correspondence between the parameter sequence s and the fixed point ζ is one-to-one, not only is the function $\psi(s) \stackrel{\text{def}}{=} (x_0, y_0)$ well-defined, but also the conjugacy relation $f \circ \psi = \psi \circ \sigma$ must be satisfied. This is because shifting the period in s forward one symbol corresponds to moving the asterisk in the system above downward to the next y-equation, i.e. $\psi\big(\sigma(s)\big) = (x_1, y_1) = f(x_0, y_0) = f\big(\psi(s)\big)$. Indeed, the homeomorphic property for the map ψ is what to be rigorously shown below by the uniform contraction mapping principle.

To do this, let $\lambda < 1$ be as in (1) and let $\mu > 1$ be such a constant that $\lambda\mu < 1$. Then it is clear that the function

$$d(\zeta, \zeta') \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \frac{1}{\mu^{|k|}} (|y_k - y'_k| + |x_{k+1} - x'_{k+1}|)$$

defines a topologically equivalent metric on $Q^{\mathbb{Z}}$, where the infinite sum is understood as the limit of $\sum_{-\ell}^{k}$ when $k \to \infty$ and $\ell \to \infty$ independently. It follows from the estimate below that $\Phi(\cdot, s)$ is contractive under this metric with a contraction constant $\lambda\mu < 1$ uniformly for $s \in \Sigma_2$.

$$\begin{aligned}
&\tfrac{1}{\mu^{|k|}}\left[|h_{2,s_k}(x_k, y_{k+1}) - h_{2,s_k}(x'_k, y'_{k+1})| + |h_{1,s_k}(x_k, y_{k+1}) - h_{1,s_k}(x'_k, y'_{k+1})|\right] \\
&\quad \le \tfrac{1}{\mu^{|k|}}\Big[\big(||D_1 h_{2,s_k}|| + ||D_1 h_{1,s_k}||\big)|x_k - x'_k| + \\
&\qquad\qquad \big(||D_2 h_{2,s_k}|| + ||D_2 h_{2,s_k}||\big)|y_{k+1} - y'_{k+1}|\Big] \\
&\quad \le \nu(k)\tfrac{|x_k - x'_k|}{\mu^{|k-1|}} + \nu(-k)\tfrac{|y_{k+1} - y'_{k+1}|}{\mu^{|k+1|}} \qquad \text{(by (1))} \\
&\quad \le \lambda\mu\Big(\tfrac{|x_k - x'_k|}{\mu^{|k-1|}} + \tfrac{|y_{k+1} - y'_{k+1}|}{\mu^{|k+1|}}\Big),
\end{aligned}$$

where $\nu(k) \stackrel{\text{def}}{=} \lambda/\mu < \lambda\mu$ if $k > 0$ and $\lambda\mu$ if $k \le 0$.

It is also easy to see that as functions mapping from $Q^{\mathbb{Z}}$ into itself $\Phi(\cdot, s)$ is also continuous in the parameter $s \in \Sigma_2$. Thus, ψ is continuous. Moreover, since the symbolic space Σ_2 and the product space $Q^{\mathbb{Z}}$ are compact and Hausdorff, ψ is also a homeomorphism. Let $\phi = \psi^{-1}$. Then ϕ is the required topological conjugacy between $\{f, \Delta\}$ and $\{\sigma, \Sigma_2\}$. It maps points of Δ to their itineraries in Σ_2.

Last, to complete the proof, we need to prove the claim. Recall that $f\big|_{H_i} = (f_{1,i}, f_{2,i})$ and $f^{-1}\big|_{V_i} = (g_{1,i}, g_{2,i})$. For simplicity of notation, we will drop the subscript i from these component functions $f_{j,i}$, $g_{j,i}$ for the remainder of the proof. To find h_2, we derive first from the relations $(\bar{x}, \bar{y}) = (f_1, f_2)(x, y)$, $(x, y) = (g_1, g_2)(\bar{x}, \bar{y})$ the identity

$$y = g_2(f_1(x, y), \bar{y}). \tag{2}$$

By the assumption we have $||D_j g_2||\ ||D_j f_1|| < \frac{1}{4} < 1$ for $j = 1,\ 2$. Thus, the implicit function theorem implies that $y = h_2(x, \bar{y})$ can be solved from the equation above locally at the point $(\bar{x}, \bar{y}) = f(x, y)$. It is also easy to see that this function can be uniquely and differentiably extended to the entire region Q. In fact, for every x and $\bar{y}$, because $f^{-1}(V \cap \{y = \bar{y}\}$ is a graph over $0 \le x \le 1$, it intersects with the line $x = x$ at a unique point (x, y), which in turn given the image $(\bar{x}, \bar{y}) = f(x, y)$. This is true because of condition(a). Let $(x, y) = f^{-1}(\bar{x}, \bar{y})$ and then the global extension for h_2 follows immediately. Having obtain this function, the other one is self-evident, namely, $h_1 \stackrel{\text{def}}{=} f_1(\cdot, h_2(\cdot, \cdot))$. To show the estimate (1), differentiate (2) with $h_2(x, \bar{y})$ substituting for y. The implicit function theorem then yields

$$\begin{aligned}
||D_1 h_1|| + ||D_1 h_2|| &\le \frac{||D_1 f_1||\big(1 + ||D_1 g_2||\big)}{1 - ||D_1 g_2||\ ||D_2 f_1||} \\
||D_2 h_1|| + ||D_2 h_2|| &\le \frac{||D_2 g_2||\big(1 + ||D_2 f_1||\big)}{1 - ||D_1 g_2||\ ||D_2 f_1||}.
\end{aligned}$$

Now a direct estimate with $||D_j f_1||$ and $||D_j g_2|| < \frac{1}{2}$ from our hypothesis (b) yields the desired inequality (1) . This completes the proof. □