[Lecture Note 3]

Hartman-Grobman Theorem

Let A be a $n \times n$ nonsingular matrix which does not have eigenvalues on the unit circle. Let E^s be the generalized eigenspace of A for eigenvalues whose moduli are less than 1 and E^u be the generalized eigenspace of A for eigenvalues whose moduli are greater than 1. Then $\mathbb{R}^n = E^s \oplus E^u$ and $x = x_s + x_u$ with $x_s \in E^s,\ x_u \in E^u$. Denote by $A_s = A|_{E^s},\ A_u = A|_{E^u}$. Then by choosing an appropriate coordinate system in $E^s,\ E^u$ by Jordan's canonical form for A, we can assume

$$|A_s| < 1,\ |A_u^{-1}| < 1 < |A_u|.$$

In fact, one can treat $x_s \in \mathbb{R}^a,\ x_u \in \mathbb{R}^b$ where $a = \dim(E^s),\ b = \dim(E^u),\ a + b = n$, $A = \operatorname{diag}(A_s, A_u)$ with A_s an $a \times a$ matrix with eigenvalues inside the unit circle, and A_u a $b \times b$ matrix with eigenvalues outside the unit circle. Also, we can use the Euclidean norm for the coordinate system x and the corresponding matrix norm for A satisfying the bounds above. Such a norm is referred to as an adapted norm for the matrix A.

Definition 1. *A fixed point x_0 of a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a hyperbolic fixed point if the linearization $Df(x_0)$ is nonsingular and has no eigenvalues on the unit circle.*

Definition 2. *Two dynamical systems $f : U \to U$ and $g : V \to V$ with $U,\ V$ open sets in $\mathbb{R}^n$ are said to be topologically conjugate if there is a homeomorphism $\phi : U \to V$ so that*

$$\phi \circ f = g \circ \phi$$

That is, the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ V & \xrightarrow[g]{} & V \end{array}$$

Theorem 1 (The Hartman-Grobman Theorem)**.** *Let x_0 be a hyperbolic fixed point of a continuously differentiable map f in $\mathbb{R}^n$. Then there is a small open neighborhood U of x_0 so that f on U is topologically conjugate to its linearization $Df(x_0)$.*

Proof. Let $|\cdot|$ be an adapted norm for $A = Df(x_0)$ with $\alpha = \max(|A_s|, |A_u^{-1}|) < 1$. By making this transformation, $x \to x - x_0$, we can assume wlog that $x_0 = 0$. Also, by extending the map globally, $f \to Df(0)x + \rho_r(x)(f(x) - Df(0)x)$, using a cut-off function ρ_r, we only need to prove the global C^0-conjugacy result for functions of the form $f(x) = Ax + h(x)$ where $A = Df(0)$, $h(0) = 0,\ Dh(0) = 0$. Because of the continuity, we can choose a small $r > 0$ so that $\sup_{\mathbb{R}^n}(|h(x)| +$

$|Dh(x)|) < \delta$ for a small $\delta > 0$ for which the Global Inverse Function Theorem applies for f for which f^{-1} is also in $C^1(\mathbb{R}^n)$.

The proof is to find a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ of the following form

$$\phi = \mathrm{id} + v$$

where $v \in X := C^0(\mathbb{R}^n)$ so that $\phi \circ (A + h) = A \circ \phi$. Denote by $|\cdot|_0$ the norm for X. It is straightforward to verify that the conjugacy equation is equivalent to either of the following fixed-point problems

$$\begin{aligned} v &= -h \circ (A+h)^{-1} + A \circ v \circ (A+h)^{-1} \\ v &= A^{-1} \circ h + A^{-1} \circ v \circ (A+h). \end{aligned}$$

We use the first equation to define v_s and the second equation to define v_u. Specifically, for any $v \in X,\ h \in Y := N_\delta(0) \subset C^1(\mathbb{R}^n)$, define $F(v,h) = F(v,h)_s + F(v,h)_u$ by the relations

$$\begin{aligned} F(v,h)_s &= -h_s \circ (A+h)^{-1} + A_s \circ v_s \circ (A+h)^{-1} \\ F(v,h)_u &= A_u^{-1} \circ h_u + A_u^{-1} \circ v_u \circ (A+h). \end{aligned}$$

It is straightforward to verify that for $v, w \in X$ and $h \in Y$,

$$\begin{aligned} &F(v,h) \in X \text{ with } |F(v,h)|_0 \le |h|_0 + \alpha |v|_0, \\ &|F(v,h) - F(w,h)|_0 \le \alpha |v - w|_0, \\ &F(0,0) = 0, \text{ and } F(v,h) \text{ is continuous in } h \in Y. \end{aligned}$$

That is, $F : X \times Y \to X$ is continuous and $F(\cdot, h)$ is a uniform contraction from X to itself. Thus, by the Uniform Contraction Principle I, there is a continuous function $v : Y \to X$ so that $F(w,h) = w$ with $(w,h) \in X \times Y$ iff $w = v(h)$. Hence, $\phi(h) = \mathrm{id} + v(h) \in X$ and $\phi \circ (A+h) = A \circ \phi$ holds.

It remains to show that $\phi(h)$ is a homeomorphism. To this end, we consider the equation $(A+h) \circ \psi = \psi \circ A$ with $\psi = \mathrm{id} + w$, $w \in X$ and $h \in Y$. The same arguments above can be used to show that there is a unique such $\psi(h)$ for each $h \in Y$, making δ smaller if necessary for $Y = N_\delta(0)$. Because of the conjugacy equations for both ϕ and ψ, we have

$$\begin{aligned} \phi(h) \circ \psi(h) &= (A^{-1} \circ \phi(h) \circ (A+h)) \circ \psi(h) \\ &= A^{-1} \circ \phi(h) \circ ((A+h) \circ \psi(h)) = A^{-1} \circ \phi(h) \circ \psi(h) \circ A. \end{aligned}$$

It can also be verified easily that

$$\phi(h) \circ \psi(h) = \mathrm{id} + w(h) + v(h) \circ [\mathrm{id} + w(h)] := \mathrm{id} + k$$

with $k \in X$. Hence,

$$k = A^{-1} \circ k \circ A \Leftrightarrow k = A \circ k \circ A^{-1}$$

implying $F(k, 0) = k$. By the uniqueness of the fixed point and the identity $F(0,0) = 0$, we conclude $k = 0$ and $\phi(h) \circ \psi(h) = \mathrm{id}$ follows. Similarly, we can show $\psi(h) \circ \phi(h) = \mathrm{id}$, showing $\phi(h)$ is indeed a homeomorphism on $\mathbb{R}^n$. □

Reference: S.-N. Chow and J.K. Hale, Methods of Bifurcation Theory, Springer-Verlag, 1982.