Consider the primal LP problem of maximizing $z = c_1x_1 + \cdots + c_nx_n$ subject to $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ for $i = 1, 2, \ldots, m$ and $x_j \geq 0$ for $j = 1, 2, \ldots, n$. Recall that the simplex method is to maximize the component z of the solution, $(z, x_1, \ldots, x_{n+m})$, to the augmented linear system of equations: $z - c_1x_1 - \cdots - c_nx_n = 0,\ a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+i} = b_i$ for $1 \leq i \leq m, x_j \geq 0, 1 \leq j \leq n+m$ with $x_{n+i} \geq 0$ being the slack variables. Recall that the simplex method is to use elementary row reductions to find a feasible echelon form of the equality system at each step so that a basic feasible solution is obtained by setting all the nonbasic variables zeros and that the corresponding basic feasible solution is to have an improved z-value. Denote the last solution form (the same at any intermediate step) as

$$\begin{aligned} z + \bar{c}_1x_1 + \cdots + \bar{c}_{n+m}x_{n+m} &= b_0^* = \textstyle\sum_{i=1}^m y_i^* b_i \\ \bar{a}_{11}x_1 + \cdots + \bar{a}_{1\ (n+m)}x_{n+m} &= b_1^* \\ &\vdots \\ \bar{a}_{m1}x_1 + \cdots + \bar{a}_{m\ (n+m)}x_{n+m} &= b_m^* \\ x_i = 0 &\text{ for } n \text{ many non-basic variables and} \\ x_j \geq 0 &\text{ for } m \text{ many basic variables.} \end{aligned}$$

This form is feasible echelon in the sense that variable x_j is a basic variable for some $1 \leq j \leq n+m$ if and only if there is an i_j so that the coefficient of x_j from the i_jth constraint equation is $\bar{a}_{i_j j} = 1$, and all other coefficients of x_j are zeros: $\bar{a}_{ij} = 0, i \neq i_j$ and $\bar{c}_j = 0$. It is the final solution form if $\bar{c}_i \geq 0$ for all i by the optimality test and $b_i^* \geq 0$ for all $1 \leq i \leq m$. The corresponding basic feasible solution is exactly $x_j = b_{i_j}^*$ for all basic variables and $x_i = 0$ for all non-basic variables x_i, with the corresponding z-value at the basic feasible point as $z = b_0^*$. It is also important to note that b_0^* is a linear combination of the original constraints right hand b_i because of the following reason. In fact, the coefficient y_i^* of the resource parameter b_i is the result of the elementary row reduction for the simplex method when a y_i^* multiple of the ith constraint row is added to the 0th row. More importantly, because the $(n+i)$th slack variable x_{n+i} has the same constant coefficient, 1, as parameter b_i does to start the simplex method, they both have the same coefficient for the z-equation after every row operation, and especially, at the final solution step, both coefficient must be equal: $\bar{c}_{n+i} = y_i^*$. By definition, the shadow price of the ith resource constraint is the rate of change of the optimal solution with respect to the ith resource parameter b_i, that is, $\frac{\partial b_0^*}{\partial b_i}$, which by the z-equation of the last solution form we find $\frac{\partial b_0^*}{\partial b_i} = y_i^* = \bar{c}_{n+i}$, the coefficient of the ith slack variable x_{n+i}.

In matrix notation, the primal LP problem is $\max z = c^T x$ sub.t. $Ax \leq b$, componentwise, with $A = A_{m\times n}$, $c = [c_1, \ldots, c_n]^T, x = [x_1, \ldots, x_n]^T, b = [b_1, \ldots, b_m]^T$ being vectors. Let $x_s = [x_{n+1}, \ldots, x_{n+m}]^T$ denote the slack variable, $X = \begin{bmatrix} x \\ x_s \end{bmatrix}$, and $\mathbf{0} = [0, 0, \ldots, 0]^T$ the zero-vector. Then the augmented LP form is: $z - c^T x + \mathbf{0}^T x_s = 0, Ax + x_s = b, X \geq \mathbf{0}$ and the problem is to find a feasible solution so that the z value of the solution is the largest. Here for two vectors u, v, $u \leq v$ means the inequality holds componentwise. Since y_i^* are the scalar multiples used by the row operations to get the z-equation, at the last step of the simplex method we must have $z - c^T x + {y^*}^T Ax + {y^*}^T x_s = {y^*}^T b$, and equivalently $z = (c^T - {y^*}^T A)x - {y^*}^T x_s + {y^*}^T b := -\bar{c}X + {y^*}^T b$. Notice that this form $z = (c^T - y^T A)x - y^T x_s + y^T b$ holds at any step of row operations when using the components of y as the constant multiples of the corresponding constraint rows and then adding those multiples to the objective row $z - c^T x + \mathbf{0}^T x_s = 0$. When evaluate at the corresponding optimal basic feasible solution X, the z-value of the optimal solution is $z = {y^*}^T b$ for which we must have $c^T - {y^*}^T A \leq \mathbf{0}^T$ and $y^* \geq \mathbf{0}$, equivalently, ${y^*}^T A \geq c^T$ or $A^T y^* \geq c, y^* \geq \mathbf{0}$. That is, y^* is a feasible point for this LP problem: $\min w = b^T y$ sub.t. $A^T y \geq c, y \geq \mathbf{0}$, which is called the *dual* LP problem of the primal LP problem. In fact, we have the following duality result.

Theorem 1. *The shadow prince y^* for the primal LP problem $\max z = c^T x$ sub.t. $Ax \leq b, x \geq \mathbf{0}$ is a solution to the dual LP problem $\min w = b^T y$ sub.t. $A^T y \geq c, y \geq \mathbf{0}$.*

Proof. The discussion preceded the theorem shows y^* is a feasible point for the dual problem. The only part remains to prove is that the shadow price solves the dual problem. Suppose not, then there is a feasible point $\bar{y}$ for the dual problem so that $A^T \bar{y} \geq c \Leftrightarrow \bar{y}^T A \geq c^T$ and $\bar{y} \geq \mathbf{0}$ but with the dual optimal value $w = b^T \bar{y}$ that is strictly smaller than the feasible value $b^T y^*$, i.e. $b^T \bar{y} < b^T y^*$. We only need to show this inequality is false. To do so, we go back to the primal LP problem. First, instead of using the same operations on the z-equation by the shadow price vector y^*, we use instead the components of $\bar{y}$ for the constant multiples of the corresponding constraint row equations and then add these multiples to the objective row to get the equivalent z-equation: $z = (c^T - \bar{y}^T A)x - \bar{y}^T x_s + \bar{y}^T b$. Next, we use the same row operations on the constraint equations of the primal problem to get the same basic feasible point X since the z-equation is never used in row operations for the constraint equations. Since the row operations do not change the solutions to the equality LP problem, in particular, not the value of the z variable, the evaluation of the new z equation at the same optimal basic feasible point X must produce the same optimal value $z = (c^T - \bar{y}^T A)x - \bar{y}^T x_s + \bar{y}^T b = {y^*}^T b$ with x and x_s consisting of the basic feasible solution X. We now show this is impossible under the assumption that $\bar{y}^T b < {y^*}^T b$. Because $(c^T - \bar{y}^T A) \leq \mathbf{0}^T$ and $-\bar{y}^T \leq \mathbf{0}$ by the feasibility condition for the dual problem and $x \geq \mathbf{0}, x_s \geq \mathbf{0}$, we surely will have $(c^T - \bar{y}^T A)x - \bar{y}^T x_s \leq 0$ and hence ${y^*}^T b \leq \bar{y}^T b$. This contradicts the assumption that $\bar{y}^T b < {y^*}^T b$. □

Notice that, the primal and the dual problem have the same optimal value ${y^*}^T b$. As an exercise, prove the dual of the dual problem is the primal problem. That is, the primal and dual problems are the dual problem of each other. Also, prove that the solution x^* to the primal problem is the shadow price for the dual problem, that is $c^T x^* = b^T y^*$.