For this notes, $x = [x_1, \ldots, x_m]^T, y = [y_1, \ldots, y_n]^T$ are mixed strategy probability (column) vectors with $x_i \geq 0$ for all i, $y_j \geq 0$ for all j, and $\sum_i x_i = 1, \sum_j y_j = 1$, with v^T for the transpose of vector v. Also, $\mathbf{1} = [1, \ldots, 1]^T$, the vector of all entries equal to 1 for an appropriate dimension depending on context. Thus, $\sum_i x_i = x^T\mathbf{1} = \mathbf{1}^T x$. Let $A = A_{m\times n} = [a_{ij}]$ be the payoff matrix for Player X against Player Y in a zero-sum game, A_j be the column vectors of the matrix A and a_i be the row vectors of A, i.e. $A = [A_1, A_2, \ldots, A_n]$ and $A^T = [a_1^T, \ldots, a_m^T]$. Then, the expected payoff per play for Player X is $E(x, y) = \sum_{i,j} a_{ij}x_i y_j$ which can be summed in two different orders, each in a dot product form: $E(x, y) = \sum_i x_i(\sum_j a_{ij}y_j) = x^T(Ay)$ and $E(x, y) = \sum_j(\sum_i a_{ij}x_i)y_j = (x^T A)y$, and both are $E(x, y) = x^T Ay$. Finally, for two vectors a, b, $a \leq b$ means the inequality holds componentwise.

The goal of the mixed game for the players is to find $(\bar{x}, \bar{y})$ such that

$$E(\bar{x}, \bar{y}) = \max_x E(x, \bar{y}) \text{ and } E(\bar{x}, \bar{y}) = \min_y E(\bar{x}, y)$$

A solution $(\bar{x}, \bar{y})$ to this problem is called an *optimal game solution* or an optimal solution or a game solution for short, and $E(\bar{x}, \bar{y})$ is called the *game value*.

Definition 1. *$(\bar{x}, \bar{y})$ is an Nash equilibrium point if for all probability mixed strategy vectors x, y,*

$$E(x, \bar{y}) \leq E(\bar{x}, \bar{y}) \leq E(\bar{x}, y).$$

Proposition 1. *Any optimal game solution is an Nash equilibrium point and vise versa.*

Proof. $E(\bar{x}, \bar{y}) = \max_x E(x, \bar{y}) \geq E(x, \bar{y})$ for any x and $E(\bar{x}, \bar{y}) = \min_y E(\bar{x}, y) \leq E(\bar{x}, y)$ for any y, showing $(\bar{x}, \bar{y})$ is an NE by definition. Conversely, let $(\bar{x}, \bar{y})$ be an NE, then $E(x, \bar{y}) \leq E(\bar{x}, \bar{y}) \leq E(\bar{x}, y)$, implying $\max_x E(x, \bar{y}) = E(\bar{x}, \bar{y}) = \min_y E(\bar{x}, y)$. The equalities hold because $\bar{x}$ and $\bar{y}$ are in the sets over which the optimizations are taken. □

Proposition 2. *The game value is unique.*

Proof. Let $(\bar{x}, \bar{y}), (x', y')$ be two optimal solutions with game values $u = E(\bar{x}, \bar{y}), v = E(x', y')$, respectively. Then by definition, $u = E(\bar{x}, \bar{y}) \leq E(\bar{x}, y') \leq E(x', y') = v$ because $(\bar{x}, \bar{y})$ is an NE for the first inequality and (x', y') is an NE for the second inequality. Since u, v are two arbitrary NEs, we have by the same argument $v \leq u$, showing $u = v$. □

Lemma 1. *Let S be the simplex defined by $w_i \geq 0$ for all i and $\sum w_i = 1$, then $\max_{w\in S} c^T w = \max_{1\leq i\leq k}\{c_i\}$. Similarly, $\min_{w\in S} c^T w = \min_{1\leq i\leq k}\{c_i\}$.*

Proof. Consider it as an LP problem to optimize $z = c^T w$ sub.t. $\sum w_i = 1$, $w_i \geq 0$. The optimal values take place at the corners point of the simplex. The corner points are e_i whose entries are all zeros except for the ith entry which is 1, and the value of the objective function at these corner points are exactly c_i. Hence, $\max c^T w = \max_i\{c_i\}$ and $\min c^T w = \min_i\{c_i\}$ respectively. □

Proposition 3. *The dual LP problem for the LP problem of $\max w = v$ sub.t. $x^T A \geq v\mathbf{1}^T, x \geq 0, \sum_i x_i = 1$ is $\min z = u$ sub.t. $Ay \leq u\mathbf{1}, y \geq 0, \sum_j y_j = 1$. Therefore, the optimal value is the same and the solution of one problem is part of the shadow price of the other.*

Theorem 1. *$(\bar{x}, \bar{y})$ is an optimal game solution with the game value $\bar{v} = E(\bar{x}, \bar{y})$ iff $(\bar{x}, \bar{v})$ is a solution to this LP problem: $\max z = u$ sub.t. (subject to) $x^T A \geq u\mathbf{1}^T, x \geq 0, \sum x_i = 1$, and $(\bar{y}, \bar{v})$ is a solution to the dual LP problem: $\min z = u$ sub.t. $Ay \leq u\mathbf{1}, y \geq 0, \sum y_j = 1$.*

Proof. Proof of the necessity condition: As an optimal game solution $\bar{v} = E(\bar{x}, \bar{y}) = \min_y E(\bar{x}, y) = \min_y(\bar{x}^T A)y = \min_j\{\bar{x}^T A_j\}$ by Lemma 1, which implies $\bar{x}^T A_j \geq \bar{v}$ for all j and equivalently $\bar{x}^T A \geq \bar{v}\mathbf{1}^T$. That is, $\bar{x}, \bar{v}$ is a basic feasible point for the LP problem $\max z = u$ sub.t. $x^T A \geq u\mathbf{1}^T$ with $x \geq 0, \sum x_i = 1$.

We claim $\bar{x}, \bar{v}$ must be an optimal solution to the LP problem. If not, there is an x' and u such that $x'^T A \geq u\mathbf{1}^T$ with $u > \bar{v} = E(\bar{x}, \bar{y})$. That is $\min_j\{x'^T A_j\} \geq u > \bar{v}$ componentwise. By Lemma 1, we have $\min_y E(x', y) = \min_y(x'^T A)y = \min_j\{x'^T A_j\} \geq u > \bar{v} = E(\bar{x}, \bar{y})$. Since $E(x', \bar{y}) \geq \min_y E(x', y) \geq u > \bar{v} = E(\bar{x}, \bar{y})$, this contradicts the property that $(\bar{x}, \bar{y})$ is an NE. This proves the necessary condition.

Conversely, because $\bar{x}, \bar{y}$ are the optimal solutions for the dual pair with the optimal value $\bar{v}$, from $\bar{x}^T A \geq \bar{v}\mathbf{1}^T$ we have $E(\bar{x}, \bar{y}) = (\bar{x}^T A)\bar{y} \geq (\bar{v}\mathbf{1}^T)y = \bar{v}(\mathbf{1}^T y) = \bar{v}$ and from $A\bar{y} \leq \bar{v}\mathbf{1}$ we have $E(\bar{x}, \bar{y}) = \bar{x}^T(A\bar{y}) \leq x^T(\bar{v}\mathbf{1}) = \bar{v}$ and hence $E(\bar{x}, \bar{y}) = \bar{v}$. Also, for any x, $E(x, \bar{y}) = x^T(A\bar{y}) \leq \bar{v} = E(\bar{x}, \bar{y})$ and for any y, $E(\bar{x}, y) = (\bar{x}^T A)y \geq \bar{v} = E(\bar{x}, \bar{y})$, showing $(\bar{x}, \bar{y})$ is an optimal game solution with the game value $\bar{v}$. □

Theorem 2. *Let $(\bar{x}, \bar{y})$ be an NE, then $E(\bar{x}, \bar{y}) = \max_x[\min_y E(x, y)]$ over the mixed strategy probability vectors and symmetrically $E(\bar{x}, \bar{y}) = \min_y[\max_x E(x, y)]$.*

Proof. Notice that the primal LP problem can be equivalently written as $x^T A \geq u\mathbf{1}^T \Leftrightarrow \min_j x^T A_j \geq u \Leftrightarrow \min_y(x^T A)y \geq u \Leftrightarrow \min_y E(x, y) \geq u$ with the largest such u. This implies $\max_x(\min_y E(x, y)) \geq \max_x u = \bar{v} = E(\bar{x}, \bar{y})$.

We claim the equality $\max_x(\min_y E(x, y)) = \max_x u$ must hold. If not, let $w(x) = \min_y E(x, y) = \min_j\{x^T A_j\} \Leftrightarrow x^T A_j \geq w(x)$ for all j and let x' have the property that $u' = w(x') = \max_x w(x)$ but $u' > \max_x u = \bar{v}$. Then $x^T A \geq w(x)\mathbf{1}^T$ for all x. In particular, $x'^T A \geq w(x')\mathbf{1}^T = u'\mathbf{1}^T$, showing (x', u') is a basic feasible point to the LP problem. Since $\bar{v}$ is the maximal value of the LP solution, we must have $u' \leq \bar{v}$, contradicting the assumption $u' > \bar{v}$. Exactly the same argument applies to the dual problem. □