Uniqueness of Center Manifold Dynamics

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearizatoin $Df(\bar{q})$. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

Let $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u = \mathbb{E}^c \times \mathbb{E}^{su}$ with $\mathbb{E}^{su} \cong \mathbb{E}^s \times \mathbb{E}^u$ based at the fixed point. For $r > 0$, denote by $\mathbb{E}^c_r = \{\{\bar{q}\} \oplus \mathbb{E}^c\} \cap \{\|p - \bar{q}\| < r\}$ the r-neighborhood of $\bar{q}$ on its center eigenspace and similarly, $\mathbb{E}^{su}_r = \{\{\bar{q}\} \oplus \mathbb{E}^{su}\} \cap \{\|p - \bar{q}\| < r\}$.

Definition 1. *A set $W^{\mathrm{c}}_{\mathrm{loc}}$ in $N_r(\bar{q})$ is called a local center-manifold of $\bar{q}$ if (a) it is invariant under f: $f(W^{\mathrm{c}}_{\mathrm{loc}}) \cap N_r(\bar{q}) \subset W^{\mathrm{c}}_{\mathrm{loc}}$, (b) it is the graph of a C^1 function $\phi_{su} : \mathbb{E}^c_r \to \mathbb{E}^{su}_r$, and (c) $W^{\mathrm{c}}_{\mathrm{loc}}$ is tangent to $\mathbb{E}^c$ at $\bar{q}$*

$$\mathbb{T}_{\bar{q}} W^{\mathrm{c}}_{\mathrm{loc}} \cong \mathbb{E}^c,$$

i.e., $D\phi_{su}(\bar{q}) \cong 0$.

Theorem 1 (Uniqueness of Center Manifold Dynamics)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a $C^{1,1}$ diffeomorphism f in $\mathbb{R}^d$. Let $W^c_{\mathrm{loc},1}$, $W^c_{\mathrm{loc},2}$ be two $C^{1,1}$ local center manifolds of $\bar{q}$. Then there is an open neighborhood V of $\bar{q}$ and a C^1 invertible map $\kappa : W^c_{\mathrm{loc},1} \cap V \to W^c_{\mathrm{loc},2} \cap V$ so that*

$$f \circ \kappa(p) = \kappa \circ f(p)$$

for all $p \in W^c_{\mathrm{loc},1} \cap V$ so long as $f(p) \in W^c_{\mathrm{loc},1} \cap V$. Furthermore, if f is a $C^{k,1}, k \geq 1$ diffeomorphism and $W^c_{\mathrm{loc},1}$, $W^c_{\mathrm{loc},2}$ are $C^{k,1}$ manifolds, then the conjugacy κ is of C^k.

Lemma 1. *Let $W^{\mathrm{c}}_{\mathrm{loc}}$ be a local center manifold of a fixed point $\bar{q}$ of a $C^{k,1},\ k \geq 1$ diffeomorphism f in $N_{r_0}(\bar{q})$. If $W^{\mathrm{c}}_{\mathrm{loc}} = \{\bar{q}\} \oplus \mathbb{E}^c_{r_0}$, then for sufficiently small $0 < r < r_0/2$, there is a $C^{k,1}$ diffeomorphism $\tilde{f}$ in $\mathbb{R}^d$ such that the following properties hold: (i) $\tilde{f}|_U = f$ with $U = N_r(\bar{q})$; (ii) $\tilde{f}|_{\{\|p-\bar{q}\| \geq 2r\}} = Df(\bar{q})$; (iii) the whole center eigenspace $\{\bar{q}\} \oplus \mathbb{E}^c$ is invariant under $\tilde{f}$; and (iv) $\|\tilde{f} - D\tilde{f}(\bar{q})\|_1 \to 0$ as $r \to 0$.*

Proof. Without loss of generality we assume the fixed $\bar{q}$ is translated to the origin and identify $\mathbb{R}^d \cong \mathbb{E}^c \times \mathbb{E}^{su}$ with $\mathbb{R}^d = \mathbb{E}^c \times \mathbb{E}^{su}$. Let $x = (x_c, x_{su}) \in \mathbb{R}^d = \mathbb{E}^c \times \mathbb{E}^{su}$ be the coordinate system for the eigenspaces splitting for which $J := Df(\bar{q}) = \mathrm{diag}(A_c, A_{su})$, $f = (f_c, f_{su})$, $f_i(x) = A_i x_i + h_i(x)$ where $h(x) = f(x) - Jx$, and $\|x\| = \|x_c\| + \|x_{su}\|$. Because $W^{\mathrm{c}}_{\mathrm{loc}} = \mathbb{E}^c_{r_0}$, we have $x \in W^{\mathrm{c}}_{\mathrm{loc}}$

iff $x_{su} = 0$ and $\|x_c\| < r_0$. Because $W^{\mathrm{c}}_{\mathrm{loc}}$ is invariant, $f(W^{\mathrm{c}}_{\mathrm{loc}}) \cap N_{r_0} = W^{\mathrm{c}}_{\mathrm{loc}}$, we have $f_{su}(x_c, 0) = 0$ for $\|x_c\| < r_0$. That is,

$$0 = f_{su}(x_c, 0) = A_{su}0 + h_{su}(x_c, 0) = h_{su}(x_c, 0).$$

Conversely, if $h_{su}(x_c, 0) = 0$ for all $x_c \in V \subset \mathbb{E}^c$ for an open set V containing 0, then V must be invariant for f, and is contained in a center-manifold of f.

Let ρ_r be a C^∞ cut-off function with $\rho_r(x) = 1$ if $\|x\| \leq r$ and $\rho_r(x) = 0$ if $\|x\| \geq 2r$. Define $\tilde{f} = (\tilde{f}_c, \tilde{f}_{su})$ as

$$\tilde{f}(x) = Jx + \tilde{h}(x), \ \text{ with } \ \tilde{h}(x) = \rho_r(x)h(x).$$

Then for $0 < r < r_0/2$, we first have $\tilde{f}|_U = f$ for $U = N_r$, i.e. $\tilde{f}$ is a global extension of f on U and $\tilde{f}|_{\{\|p-\bar{q}\|\geq 2r\}} = J$.

Next for $x = (x_c, 0) \in W^{\mathrm{c}}_{\mathrm{loc}}, \ \|x_c\| < 2r < r_0$ we have

$$\tilde{f}_{su}(x_c, 0) = A_{su}0 + \rho_r(x_c, 0)h_{su}(x_c, 0) = \rho_r(x_c, 0) \cdot 0 = 0.$$

In addition, for $x = (x_c, 0)$ and $\|x_c\| \geq 2r$, we have

$$\tilde{f}_{su}(x_c, 0) = A_{su}0 + \rho_r(x_c, 0)h_{su}(x_c, 0) = 0 \cdot h_{su}(x_c, 0) = 0.$$

This implies the whole center eigenspace $\mathbb{E}^c$ is invariant for $\tilde{f}$.

Last, because $\tilde{f}|_U = f$, we have $D\tilde{f}(0) = Df(0) = J$ and $\tilde{f}(x) - D\tilde{f}(0)x = \rho_r(x)(f(x) - Df(0)x)$, implying $\|\tilde{f} - D\tilde{f}(\bar{q})\|_1 \to 0$ if $r \to 0$, which also implies $\tilde{f}$ is globally invertible for small r. ☐

Lemma 2. *Let $W^{\mathrm{c}}_{\mathrm{loc}}$ be a $C^{k,1}, k \geq 1$ local center manifold of a fixed point $\bar{q}$ of a $C^{k,1}$ diffeomorphism f in $N_{r_0}(\bar{q})$. Then for sufficiently small $0 < r < r_0/2$, there is a $C^{k,1}$ local center-stable manifold $W^{\mathrm{cs}}_{\mathrm{loc}}$ and a $C^{k,1}$ local center-unstable manifold $W^{\mathrm{cu}}_{\mathrm{loc}}$ in $N_r(\bar{q})$ so that $W^{\mathrm{c}}_{\mathrm{loc}} \cap N_r(\bar{q}) = W^{\mathrm{cs}}_{\mathrm{loc}} \cap W^{\mathrm{cu}}_{\mathrm{loc}} \cap N_r(\bar{q})$. Moreover, $W^{\mathrm{cs}}_{\mathrm{loc}}$ is equipped with a C^k stable foliation and $W^{\mathrm{cu}}_{\mathrm{loc}}$ is equipped with a C^k unstable foliation.*

Proof. Use the same coordinate system setup as in the proof of Lemma 1 above. Let ρ_r be the same type of cut-off function as well. Let $x_{su} = \phi_{su}(x_c), x_c \in \mathbb{E}^c_{r_0}$ be the $C^{k,1}$ function for $W^{\mathrm{c}}_{\mathrm{loc}}$. Define a change of variables in $\mathbb{R}^d$, $y = g(x)$ as below

$$\begin{cases} y_c = x_c \\ y_{su} = x_{su} - \rho_{r_0}(x_c)\phi_{su}(x_c), \end{cases}$$

whose inverse, $x = g^{-1}(y)$ is explicitly

$$\begin{cases} x_c = y_c \\ x_{su} = y_{su} + \rho_{r_0}(y_c)\phi_{su}(y_c). \end{cases}$$

Because ϕ_{su} is $C^{k,1}$, so is g and g^{-1}. Let $\bar{f}(y) = g \circ f \circ g^{-1}$, which is $C^{k,1}$ as well. Then, g transforms f's local center manifold $W^{\mathrm{c}}_{\mathrm{loc}} = \mathrm{graph}(\phi_{su})$ to the flat local center manifold $\mathbb{E}^c_{r_0} = \{y_{su} = 0\} \cap N_{r_0}$ for $\bar{f}$. By Lemma 1, let $\tilde{f}$ be the extension of $\bar{f}$. Since $\|\tilde{f} - D\tilde{f}(\bar{q})\|_1 \to 0$ as $r \to 0$, the Center-Stable Manifold Theorem, the Center-Unstable Manifold Theorem, the Stable-Foliation Theorem, and the Unstable-Foliation Theorem all apply. Let W^{cs}, W^{cu} denote the center-stable manifold, the center-unstable manifold, respectively. Because $\mathbb{E}^c$ is invariant for $\tilde{f}$ whose restricted dynamics on it and outside the unbounded region $\{\|p - \bar{q}\| \geq 2r\}$ is the linear map A_c, $\tilde{f}$ cannot grow in either directions of iteration faster than any geometric rate. Therefore, by the definition and uniqueness of both W^{cs} and W^{cu} for $\tilde{f}$, $\mathbb{E}^c$ must be contained by both manifolds. Transform back these manifolds by g^{-1} and restrict the map $\bar{f} = g^{-1} \circ \tilde{f} \circ g$ in a small neighborhood $N_r(\bar{q})$ to recover the required structures for $f = \bar{f}|_U$ with $U = N_r(\bar{q})$. In particular, $W^{\text{cs}}_{\text{loc}}$, $W^{\text{cu}}_{\text{loc}}$ are $C^{k,1}$, both containing $W^{\text{c}}_{\text{loc}}$, and their foliations, $\mathcal{F}^s$, $\mathcal{F}^u$ are C^k. This completes the proof. □

Proof of Theorem 1. We prove first a special case for which $W^c_{\text{loc},1}$, $W^c_{\text{loc},2}$ both lie on one local center-stable manifold $W^{\text{cs}}_{\text{loc}}$ equipped with a C^k stable foliation $\mathcal{F}^s$, or on one local center-unstable manifold $W^{\text{cu}}_{\text{loc}}$ equipped with a C^k unstable foliation $\mathcal{F}^u$. Since the proof for the latter is the same as for the former, differing only by considering the inverse of f, we only consider the $W^{\text{cs}}_{\text{loc}}$ case.

Because both $W^c_{\text{loc},1}$ and $W^c_{\text{loc},2}$ are tangent to $\mathbb{E}^c$ at $\bar{q}$, and the stable foliation $\mathcal{F}^s(\bar{q})$ is tangent to $\mathbb{E}^s$ at $\bar{q}$, for small enough neighborhood $N_r(\bar{q})$, the foliation fibers $\mathcal{F}^s(p)$ intersect both $W^c_{\text{loc},1}$ and $W^c_{\text{loc},2}$ transversely. The conjugacy κ is defined as follows. For any point $p \in W^c_{\text{loc},1}$, the foliation $\mathcal{F}^s(p)$ through p has a unique intersection with $q \in W^c_{\text{loc},2}$, denote it by $q = \kappa(p)$. Because the foliation is C^k and the intersection is transversal, κ is C^k as well. Moreover, it can be seen easily that κ is invertible because $\mathcal{F}^s$ defines an equivalence relation on $W^{\text{cs}}_{\text{loc}}$ and its intersections with both local center manifolds are unique and transversal. Furthermore,κ commutes with f because whenever it is defined, wherever $\mathcal{F}^s(f(p))$ or $\mathcal{F}^s(f^{-1}(p))$ intersects $W^c_{\text{loc},2}$ is wherever $f(\mathcal{F}^s(p))$ or $f^{-1}(\mathcal{F}^s(p))$ intersects $W^c_{\text{loc},2}$ by the invariance of $\mathcal{F}^s$ under f, showing $\kappa \circ f(p) = f \circ \kappa(p)$. This proves the special case.

Next, let $W^c_{\text{loc},1}$, $W^c_{\text{loc},2}$ be any two $C^{k,1}$ local center manifolds of $\bar{q}$. Apply Lemma 2 to obtain a $C^{k,1}$ local center-stable manifold $W^{\text{cs}}_{\text{loc},1}$ containing $W^c_{\text{loc},1}$, together with a C^k stable foliation for $W^{\text{cs}}_{\text{loc},1}$. Similarly, apply Lemma 2 to obtain a $C^{k,1}$ local center-unstable manifold $W^{\text{cu}}_{\text{loc},2}$ containing $W^c_{\text{loc},2}$, together with a C^k unstable foliation for $W^{\text{cu}}_{\text{loc},2}$. Let

$$W^c_{\text{loc},3} = W^{\text{cs}}_{\text{loc},1} \cap W^{\text{cu}}_{\text{loc},2} \cap N_r(\bar{q}).$$

Then by the Local Center Manifold Theorem, it is a $C^{k,1}$ local center manifold. Since the dynamics of f on $W^c_{\text{loc},1}$ is C^k conjugate to $W^c_{\text{loc},3}$ for both being on $W^{\text{cs}}_{\text{loc},1}$ by the special case, and similarly, f on $W^c_{\text{loc},2}$ is C^k conjugate to $W^c_{\text{loc},3}$ for both being on $W^{\text{cu}}_{\text{loc},2}$, f on $W^c_{\text{loc},1}$ is hence C^k conjugate to $W^c_{\text{loc},2}$ by conjugacy's transitivity. This proves the theorem. □