Stable and Unstable Foliations

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, counting multiplicity. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

Definition 1. *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and $\alpha,\ \beta$ be any constants satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1 < \beta < \min\{|\sigma^u|\}.$$

Let $W^{\mathrm{cs}} = \{p : \sup\{\beta^{-n}[f^n(p) - \bar{q}] : n \geq 0\} < \infty\}$ be the center-stable manifold of $\bar{q}$. For every $q \in W^{\mathrm{cs}}$ the stable-fiber of q is defined as

$$\mathcal{F}^s(q) = \{p \in W^{\mathrm{cs}} : \sup\{\alpha^{-n}[f^n(p) - f^n(q)] : n \geq 0\} < \infty\}$$

and the collection

$$\mathcal{F}^s = \{\mathcal{F}^s(q) : q \in W^{\mathrm{cs}}\}$$

is called the stable-foliation of W^{cs}.

Notice that the stable-fiber defines an equivalence relation on W^{cs}: $q \in \mathcal{F}^s(q)$; $p \in \mathcal{F}^s(q)$ iff $q \in \mathcal{F}^s(p)$ and $\mathcal{F}^s(q) = \mathcal{F}^s(p)$. Also, the foliation is an invariant family with $f(\mathcal{F}^s(q)) = \mathcal{F}^s(f(q))$, and W^{cs} can be filled by fibers through a center manifold as a stem

$$f(\mathcal{F}^s(q)) = \mathcal{F}^s(f(q)), \quad W^{\mathrm{cs}} = \cup_{q \in W^{\mathrm{c}}} \mathcal{F}^s(q).$$

In addition, the stable manifold is the fiber through $\bar{q}$, $W^{\mathrm{s}} = \mathcal{F}^s(\bar{q})$, see Fig.1.

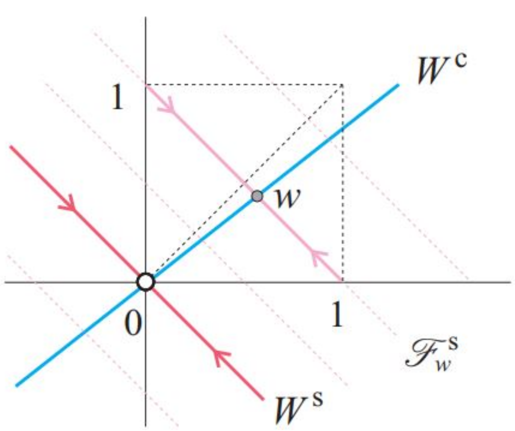

Figure 1. The dynamics of the transition matrix of a Markov process at the trivial fixed point 0 is captured by its foliation through W^{c} which is spanned by the steady-state distribution vector w.

A function f is of $C^{k,1}$ if f is in C^k (itself and all derivatives up to order k are uniformly continuous and bounded in $\mathbb{R}^d$) and its kth derivative is globally Lipschitz continuous. We will use $\|f\|_k$ to denote its C^k norm.

Theorem 1 (Stable Foliation Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a $C^{1,1}$ diffeomorphism f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u = \mathbb{E}^{cs} \times \mathbb{E}^u$ based at the fixed point. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies there is a C^1 function*

$$\psi_{cu} = (\psi_c, \psi_u) : \mathbb{E}^{cs} \times \mathbb{E}^s \to \mathbb{E}^c \times \mathbb{E}^u$$

such that

(i) $q = (q_{cs}, q_u) \in W^{\rm cs}$ *iff* $q_u = \psi_u(q_{cs}, q_s)$ *with* $q_{cs} = (q_s, q_c)$*, i.e.,*

$$W^{\rm cs} = \mathrm{graph}(\phi_u) \;\; \textit{with } \phi_u(q_{cs}) = \psi_u(q_{cs}, q_s).$$

(ii) $\mathcal{F}^s(q) = \mathrm{graph}(\psi_{cu}(q_{cs}, \cdot))$ *for* $q \in W^{\rm cs}$*, i.e.,*

$$p = (p_s, p_c, p_u) \in \mathcal{F}^s(q) \;\; \textit{if and only if } (p_c, p_u) = \psi_{cu}(q_{cs}, p_s).$$

(iii) f *is a contraction on each* $\mathcal{F}^s(q)$ *uniformly for all* $q \in W^{\rm cs}$.

(iv) $\mathcal{F}^s(\bar{q})$ *coincides with the stable manifold* $\mathcal{F}^s(\bar{q}) = W^{\rm s}$ *and*

$$\mathbb{T}_{\bar{q}}\mathcal{F}^s(\bar{q}) \cong \mathbb{E}^s.$$

(v) If f *is* $C^{k,1}, k \geq 1$*, then* ψ_{cu} *is* C^k.

(vi) $\mathcal{F}^s$ *is independent of any two different choices in* α.

Proof. Let $\lambda_1 = \max\{|\sigma^{cs}|\} = 1$ and $\lambda_2 = \min\{|\sigma^u|\} > 1$. Let $\mu_1 = \max\{|\sigma^s|\} < 1 = \min\{|\sigma^{cu}|\} = \mu_2$. Then both $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$ are pseudo-hyperbolic splits for J, and the μ-split is a sub-tight split to the λ-split because $\mu_2 = 1 = \lambda_1$ implies $\mu_1\lambda_1 < \mu_2$ automatically. In addition, ${\lambda_1}^k = 1 < \lambda_2$, ${\mu_1}^k \leq \mu_1 < 1 = \mu_2$, and $\mu_1{\lambda_1}^k = \mu_1 < \mu_2$ hold for any $k \geq 1$. Therefore, the result follows from the Sub-Foliation of Left-Manifold Theorem. □

By applying the theorem to f^{-1} one can obtain the Unstable Foliation Theorem for the Center-Unstable Manifold of $\bar{q}$.