Center Manifold Theorem

Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma^s = \sigma(J) \cap \{|z| < 1\}, \sigma^c = \sigma(J) \cap \{|z| = 1\}, \text{ and } \sigma^u = \sigma(J) \cap \{|z| > 1\}$$

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearizatoin $Df(\bar{q})$. Let

$$\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.$$

Definition 1. *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and β be any constant satisfying*

$$1 < \beta < \min\{|\sigma^u|\}.$$

The center-stable manifold of the fixed point $\bar{q}$ for f is

$$W^{\mathrm{cs}} = \{p : \{\beta^{-n}[f^n(p) - \bar{q}]\}_{n=0}^{\infty} \textit{ is a bounded sequence}\}.$$

Theorem 1 (Center-Stable Manifold Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^{cs} \times \mathbb{E}^u$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies W^{cs} is independent of any two different choices in β. Also, W^{cs} is the graph of a C^1 function $\phi_u : \mathbb{E}^{cs} \to \mathbb{E}^u$*

$$W^{\mathrm{cs}} = \mathrm{graph}(\phi_u),$$

and the tangent space of W^{cs} at the fixed point is the center-stable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{cs}} \cong \mathbb{E}^{cs}.$$

Furthermore, if $f \in C^k(\mathbb{R}^d)$, $1 \le k < \infty$, then $\phi_u \in C^k(\mathbb{E}^{cs}, \mathbb{E}^u)$, and if $f \in C^{k,1}(\mathbb{R}^d)$, then $\phi_u \in C^{k,1}(\mathbb{E}^{cs}, \mathbb{E}^u)$.

Proof. Let $\lambda_1 = \max\{|\sigma^{cs}|\} = 1$ and $\lambda_2 = \min\{|\sigma^u|\} > 1$. Then $[\lambda_1, \lambda_2]$ is a pseudo-hyperbolic split for J. In addition, the condition ${\lambda_1}^k = 1 < \lambda_2$ holds automatically for any $k \ge 1$. Hence, the result follows from the λ-Left Manifold Theorem. $\square$

Definition 2. *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and α be any constant satisfying*

$$\max\{|\sigma^s|\} < \alpha < 1.$$

The center-unstable manifold of the fixed point $\bar{q}$ is

$$W^{\mathrm{cu}} = \{p : \{\alpha^n[f^{-n}(p) - \bar{q}]\}_{n=0}^{\infty} \textit{ is a bounded sequence}\}.$$

By applying Theorem 1 to f^{-1} we obtain the following result.

Theorem 2 (Center-Unstable Manifold Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ with splitting $\mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^{cu}$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies W^{cu} is independent of any two different choices in α. Also, W^{cu} is the graph of a C^1 function $\phi_s : \mathbb{E}^{cu} \to \mathbb{E}^s$*

$$W^{\mathrm{cu}} = \mathrm{graph}(\phi_s),$$

and the tangent space of W^{cu} at the fixed point is the center-unstable eigenspace

$$\mathbb{T}_{\bar{q}} W^{\mathrm{cu}} \cong \mathbb{E}^{cu}.$$

Furthermore, if $f \in C^k(\mathbb{R}^d)$, $1 \leq k < \infty$, then $\phi_s \in C^k(\mathbb{E}^{cu}, \mathbb{E}^s)$, and if $f \in C^{k,1}(\mathbb{R}^d)$, then $\phi_s \in C^{k,1}(\mathbb{E}^{cu}, \mathbb{E}^s)$.

Theorem 3 (Local Center-stable and Local Center-unstable Manifold Theorem)**.** *Let $\bar{q}$ be a nonhyperbolic fixed point of a diffeomorphism f in $\mathbb{R}^d$ and let $\mathbb{E}^{cs}$, $\mathbb{E}^{cu}$, $\mathbb{E}^s$, $\mathbb{E}^u$ be the center-stable, center-unstable, stable, unstable eigenspace, respectively, at $\bar{q}$ for the linearization $Df(\bar{q})$. Then there is a small neighborhood $N_r(\bar{q})$ and two differentiable functions $\phi_u : N_r(\bar{q}) \cap \mathbb{E}^{cs} \to \mathbb{E}^u$, $\phi_s : N_r(\bar{q}) \cap \mathbb{E}^{cu} \to \mathbb{E}^s$, so that the local center-stable and local center-unstable manifolds*

$$W^{\mathrm{cs}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_u), \quad W^{\mathrm{cu}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_s)$$

satisfy the following properties

(i) $W^{\mathrm{cs}}_{\mathrm{loc}}$ *contains all bounded forward orbits in* N_r.

(ii) $W^{\mathrm{cu}}_{\mathrm{loc}}$ *contains all bounded backward orbits in* N_r.

(iii) They are locally invariant, i.e., $f(W^i_{\mathrm{loc}}) \cap N_r \subseteq W^i_{\mathrm{loc}}$, $f^{-1}(W^i_{\mathrm{loc}}) \cap N_r \subseteq W^i_{\mathrm{loc}}$, $i = \mathrm{cs},\ \mathrm{cu}$

(iv) $\mathbb{T}_{\bar{q}} W^{\mathrm{cs}}_{\mathrm{loc}} \cong \mathbb{E}^{cs}$, $\mathbb{T}_{\bar{q}} W^{\mathrm{cu}}_{\mathrm{loc}} \cong \mathbb{E}^{cu}$.

Moreover, if f is C^k, $1 \leq k < \infty$, then both ϕ_u and ϕ_s are C^k, and if f is $C^{k,1}$, then both ϕ_u and ϕ_s are $C^{k,1}$.

Proof. Modify the map f by a C^∞ cut-off function $\rho_r(p - \bar{q})$ to $f \to f(p) = Df(\bar{q})p + \rho_r(p - \bar{q})(f(p) - Df(\bar{q})p)$. Then for sufficiently small r, Theorems 1 and 2 can be applied to the modified map to obtain the maps ϕ_u, ϕ_s. Restrict both to the neighborhood $N_r(\bar{q})$, then the results follow from the theorems. □

By applying the theorem above we obtain

Theorem 4 (Local Center Manifold Theorem)**.** *Let $\bar{q}$ be a nonsingular fixed point of a continuously differentiable map f in $\mathbb{R}^d$ and let $\mathbb{E}^s$, $\mathbb{E}^c$, $\mathbb{E}^u$ be the stable, center, unstable eigenspace, respectively, at $\bar{q}$ for the linearization $Df(\bar{q})$. Then there is a small neighborhood $N_r(\bar{q})$ and a differentiable function $\phi_{su} : N_r(\bar{q}) \cap \mathbb{E}^c \to \mathbb{E}^s \times \mathbb{E}^u$, so that the local center manifold*

$$W^{\mathrm{c}}_{\mathrm{loc}}(\bar{q}) := \mathrm{graph}(\phi_{su})$$

satisfies the following properties

(i) $W^{\mathrm{c}}_{\mathrm{loc}}$ *contains all orbits bounded in both forward and backward directions in* N_r.

(ii) *Every point not from* $W^{\mathrm{c}}_{\mathrm{loc}}$ *escapes* $N_r(\bar{q})$ *in either forward or backward iteration.*

(iii) *It is locally invariant,* $f(W^c_{\mathrm{loc}}) \cap N_r \subseteq W^c_{\mathrm{loc}}$, $f^{-1}(W^c_{\mathrm{loc}}) \cap N_r \subseteq W^c_{\mathrm{loc}}$.

(iv) $\mathbb{T}_{\bar{q}} W^{\mathrm{c}}_{\mathrm{loc}} \cong \mathbb{E}^c$.

Moreover, if f *is* C^k, $1 \le k < \infty$, *then* ϕ_{su} *is* C^k, *and if* f *is* $C^{k,1}$, *then* ϕ_{su} *is* $C^{k,1}$.

Proof. Let $W^{\mathrm{cs}}_{\mathrm{loc}} = \mathrm{graph}(\phi_u)$ and $W^{\mathrm{cu}}_{\mathrm{loc}} = \mathrm{graph}(\phi_s)$ be a local center-stable manifold and a local center-unstable manifold, respectively, by the previous theorem. Define

$$W^{\mathrm{c}}_{\mathrm{loc}} = W^{\mathrm{cs}}_{\mathrm{loc}} \cap W^{\mathrm{cu}}_{\mathrm{loc}}.$$

Then property (i) through (iii) follow immediately. To show the existence of ϕ_{su} and (iv), let $p = (x, y, z)$ be a coordinate system for the splitting $\mathbb{R}^d = \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u$. Then a point $(x, y, z) \in W^{\mathrm{c}}_{\mathrm{loc}}$ iff it satisfies the equations below

$$\begin{cases} x = \phi_s(y, z) \\ z = \phi_u(x, y) \end{cases} \tag{1}$$

which in turn is equivalent to $F(x, y, z) = (F_1, F_2)(x, y, z) = 0$ with

$$F_1(x, y, z) = x - \phi_s(y, z), \quad \text{and} \quad F_2(x, y, z) = z - \phi_u(x, y).$$

Obviously, the fixed point, $\bar{q} \sim (0, 0, 0)$, is a solution, $F(0, 0, 0) = 0$. Also,

$$D_{(x,z)} F(0, 0, 0) = I$$

the identity matrix in $\mathbb{R}^{d_s + d_u} \cong \mathbb{E}^s \times \mathbb{E}^u$, because $D\phi_u(0, 0) = 0$ and $D\phi_s(0, 0) = 0$. Therefore, by the Implicit Function Theorem, equation (1), i.e. $F(x, y, z) = 0$, can be solved locally as a function $\phi_{su} : N_r \cap \mathbb{E}^c \to \mathbb{E}^s \times \mathbb{E}^u$, making r smaller if necessary, so that $(x, z) = \phi_{su}(y)$ and

$$W^{\mathrm{c}}_{\mathrm{loc}} = \mathrm{graph}(\phi_{su})$$

follows. It can be directly checked that $\phi_{su}(0) = (0, 0)$ and

$$D\phi_{su}(0) = 0$$

by IFT since $D_y F(0, 0, 0) = 0$, showing property (iv). Last, that f is C^k, or $C^{k,1}$, $1 \le k < \infty$, implies ϕ_u, ϕ_s are C^k, or $C^{k,1}$, which in turn by IFT implies ϕ_{su} is C^k, or $C^{k,1}$. This completes the proof. □

The conclusion is all interesting dynamics near a nonhyperbolic fixed point of a diffeomorphism takes place on a center manifold.

Local center manifolds are not unique in general (see Fig.1), but the center manifold dynamics is in the sense that the dynamics on any two local center manifolds are smoothly conjugate. Specifically, we have the following theorem.

Theorem 5 (Uniqueness of Center Manifold Dynamics for Flow [1]). *Let $\bar{q} = 0$ be a nonhyperbolic equilibrium point of the differential equation*

$$\dot{x} = Ax + h(x)$$

where $x \in \mathbb{R}^d,\ h(0) = 0,\ Dh(0) = 0$, and h is $C^{k+1,1}, k \geq 0$. Let f be the time-1 map of the solution, $f(x) = \varphi(1, x)$ where $\varphi(t, x_0)$ is the solution of the equation with initial condition $\varphi(0, x_0) = x_0$. Let $W^c_{\text{loc},1}$, $W^c_{\text{loc},2}$ be two local center manifolds of $\bar{q}$ for f. Then there is an open neighborhood V of $\bar{q}$ and a C^k invertible map $\kappa : W^c_{\text{loc},1} \cap V \to W^c_{\text{loc},2} \cap V$ so that

$$f \circ \kappa(p) = \kappa \circ f(p)$$

for all $p \in W^c_{\text{loc},1} \cap V$ so long as $f(p) \in W^c_{\text{loc},1} \cap V$.

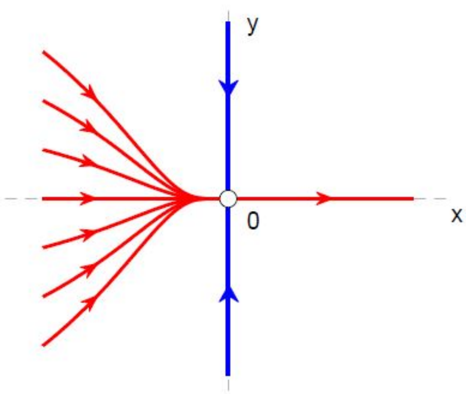

Figure 1. The phase diagram for the system of differential equations $x' = x^2,\ y' = -y$. Every red curve on the left coupled with the right x-axis is a local center manifold of the time-1 map of the solution operator at the fixed point 0. There are infinitely many local center manifolds of the origin.

Reference: 1. A. Burchard, B. Deng, and K. Lu, *Smooth conjugacy of centre manifolds*, Procedings of the Royal Society of Edingurgh, 120A, pp.61–77, 1992.