

Sub-Foliation of Left Manifold

The Sub-Foliation of Left-Manifold Theorem obtained in this note can be used to obtain all commonly encountered invariant foliations for both diffeomorphisms and ordinary differential equations. They include: the strong-stable foliation of the stable manifold, $\mathcal{F}_{\text{loc}}^{ss} \subset W_{\text{loc}}^s$, the strong-stable foliation of the center-stable manifold, $\mathcal{F}_{\text{loc}}^{ss} \subset W_{\text{loc}}^{cs}$, the stable foliation of the center-stable manifold, $\mathcal{F}_{\text{loc}}^s \subset W_{\text{loc}}^{cs}$, the unstable foliation of the center-unstable manifold, $\mathcal{F}_{\text{loc}}^u \subset W_{\text{loc}}^{cu}$, the strong-unstable foliation of the center-unstable manifold, $\mathcal{F}_{\text{loc}}^{uu} \subset W_{\text{loc}}^{cu}$, and the strong-unstable foliation of the unstable manifold, $\mathcal{F}_{\text{loc}}^{uu} \subset W_{\text{loc}}^u$. By using the stable foliation of the center-stable manifold and the unstable foliation of the center-unstable manifold, one can prove the uniqueness of center-manifold dynamics.

Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$.

Definition 1. Let $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$ be two pseudo-hyperbolic splits for J . The μ -split is called a sub-tight split if

$$(i) \quad [\mu_1, \mu_2] \leq [\lambda_1, \lambda_2], \text{ i.e., } \mu_2 \leq \lambda_1.$$

$$(ii) \quad \mu_1 \lambda_1 < \mu_2.$$

The λ -split is called a sup-tight split if (i) holds and $\lambda_1 < \lambda_2 \mu_2$.

Denote by $\mathbb{E}^{\lambda_1}$ the generalized eigenspace of J for eigenvalues $\sigma^1 = \{\lambda \in \sigma : |\lambda| \leq \lambda_1\}$ and $\mathbb{E}^{\lambda_2}$ the generalized eigenspace of J for eigenvalues $\sigma^2 = \{\lambda \in \sigma : |\lambda| \geq \lambda_2\}$. Then $\mathbb{R}^d \cong \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$. Similarly, we have $\mathbb{R}^d \cong \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$, with $\mathbb{E}^{\mu_1} \subset \mathbb{E}^{\lambda_1}$ and $\mathbb{E}^{\lambda_2} \subset \mathbb{E}^{\mu_2}$. For point $p = (x, y) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ we will use $\pi_{\mu_1} : \mathbb{R}^d \rightarrow \mathbb{E}^{\mu_1}$ for the coordinate projection map with $\pi_{\mu_1}(p) = x$. Similarly, we have $\pi_{\mu_2} : \mathbb{R}^d \rightarrow \mathbb{E}^{\mu_2}$ with $\pi_{\mu_2}(p) = y$. Also, for the λ -splitting, we have for $q = (u, w) \in \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$, $\pi_{\lambda_1} : \mathbb{R}^d \rightarrow \mathbb{E}^{\lambda_1}$ with $\pi_{\lambda_1}(q) = u$ and $\pi_{\lambda_2} : \mathbb{R}^d \rightarrow \mathbb{E}^{\lambda_2}$ with $\pi_{\lambda_2}(q) = w$. We will use exclusively $p = (x, y)$ for μ -splitted points and q for λ -splitted points. We will write $p = \pi_{\lambda}(p) = (\pi_{\lambda_1}(p), \pi_{\lambda_2}(p)) \in \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$ if we want to resolve p in the λ -splitting, and $q = \pi_{\mu}(q) = (\pi_{\mu_1}(q), \pi_{\mu_2}(q)) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ if we want to resolve q in the μ -splitting. Also, we can use π_i for subspace of $\mathbb{R}^d$ that contains $\mathbb{E}^i$. For example, $\mathbb{E}^{\mu_1}$ is a subspace of $\mathbb{E}^{\lambda_1}$, so if we let $\mathbb{F}$ be any transversal complement of $\mathbb{E}^{\mu_1}$ in $\mathbb{E}^{\lambda_1}$, then for any $u \in \mathbb{E}^{\lambda_1} \cong \mathbb{E}^{\mu_1} \times \mathbb{F}$, $\pi_{\mu_1}(u) \in \mathbb{E}^{\mu_1}$ is perfectly defined. Similarly, since $\mathbb{E}^{\lambda_2} \subset \mathbb{E}^{\mu_2}$, for any $y \in \mathbb{E}^{\mu_2}$, $\pi_{\lambda_2}(y) \in \mathbb{E}^{\lambda_2}$ is well-defined by the same reason.

Definition 2. Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $[\lambda_1, \lambda_2]$ be a pseudo-hyperbolic splits for $J = Df(\bar{q})$ and $[\mu_1, \mu_2]$ be a sub-tight split of $[\lambda_1, \lambda_2]$. Let α, β be any constants satisfying

$$\mu_1 < \alpha < \mu_2 \leq \lambda_1 < \beta < \lambda_2.$$

Let $W^{\lambda_1} = \{p : \sup\{\beta^{-n}[f^n(p) - \bar{q}] : n \geq 0\} < \infty\}$ be the lambda-left manifold of $\bar{q}$. For every $q \in W^{\lambda_1}$ the sub-fiber of q is defined as

$$\mathcal{F}^\mu(q) = \{p \in W^{\lambda_1} : \sup\{\alpha^{-n}[f^n(p) - f^n(q)] : n \geq 0\} < \infty\}$$

and the collection

$$\mathcal{F}^\mu = \{\mathcal{F}^\mu(q) : q \in W^{\lambda_1}\}$$

is called the sub-foliation of W^{λ_1} .

Notice that the sub-fiber defines an equivalence relation on W^{λ_1} : $q \in \mathcal{F}^\mu(q)$; $p \in \mathcal{F}^\mu(q)$ iff $q \in \mathcal{F}^\mu(p)$ and $\mathcal{F}^\mu(q) = \mathcal{F}^\mu(p)$. Also, the foliation is an invariant family with

$$f(\mathcal{F}^\mu(q)) = \mathcal{F}^\mu(f(q)).$$

Also W^{λ_1} can be filled by fibers through an invariant sub-manifold of W^{λ_1} as a stem that runs transverse to $\mathcal{F}^\mu$. In addition, the mu-left manifold is the fiber through $\bar{q}$, $W^{\mu_1} = \mathcal{F}^\mu(\bar{q})$.

Theorem 1 (Sub-Foliation of Left-Manifold Theorem). *Let $\bar{q}$ be a fixed point of a $C^{1,1}$ diffeomorphism f . Let $[\lambda_1, \lambda_2]$ be a pseudo-hyperbolic splits for $J = Df(\bar{q})$ and $[\mu_1, \mu_2]$ be a sub-tight split of $[\lambda_1, \lambda_2]$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies there is a C^1 function*

$$\psi_2 : \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \rightarrow \mathbb{E}^{\mu_2}$$

such that

(i) $q = (u, w) \in W^{\lambda_1}$ iff $w = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u))$, i.e.,

$$W^{\lambda_1} = \text{graph}(\phi_2) \text{ with } w = \phi_2(u) = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u)).$$

(ii) $\mathcal{F}^\mu(q) = \text{graph}(\psi_2(u, \cdot))$ for $q = (u, w) \in W^{\lambda_1}$, i.e.,

$$p = (x, y) \in \mathcal{F}^\mu(q) \text{ iff } y = \psi_2(u, x).$$

(iii) f is Lipschitz on each $\mathcal{F}^\mu(q)$ and for an adapted norm the Lipschitz constant is $\leq \alpha$ uniformly for all $q \in W^{\lambda_1}$.

(iv) $\mathcal{F}^\mu(\bar{q})$ coincides with the μ -left manifold $\mathcal{F}^\mu(\bar{q}) = W^{\mu_1}$ and

$$\mathbb{T}_{\bar{q}}\mathcal{F}^\mu(\bar{q}) \cong \mathbb{E}^{\mu_1}.$$

(v) If f is $C^{k,1}$, $k \geq 1$, $\lambda_1^k < \lambda_2$, $\mu_1^k < \mu_2$, and $\mu_1\lambda_1^k < \mu_2$, then ψ_2 is C^k .

(vi) $\mathcal{F}^\mu$ is independent of any two different choices in α .

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the sub-foliation function ψ_2 as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas. Before doing so, we first recall a few important properties about W^{λ_1} in the statements below from the proof for the Left-Manifold Theorem, assuming the fixed $\bar{q}$ is translated to $0 \in \mathbb{R}^d$.

Proposition 1. *For any $\lambda_1 < \beta < \lambda_2$, let S_β be a Banach space defined by*

$$S_\beta := \{\gamma = \{p_n\}_{n=0}^\infty : p_n \in \mathbb{R}^d, \sup\{\beta^{-n}\|p_n\| : n \geq 0\} < \infty\}$$

with norm

$$\|\gamma\|_\beta = \sup\{\beta^{-n}\|p_n\| : n \geq 0\}.$$

For any sufficiently small $\|f - Df(\bar{q})\|_1$, the orbit $\gamma_q = \{f^n(q)\}_{n=0}^\infty$ of any point $q = q_0 = (u, w) \in W^{\lambda_1}$ can be expressed as a function $\gamma_q = \gamma^(u)$ for $u \in \mathbb{E}^{\lambda_1}$ so that $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$ if $f \in C^k(\mathbb{R}^d)$ and $\lambda_1^k < \lambda_2$. Moreover, for any $u, u' \in \mathbb{E}^{\lambda_1}$*

$$\|\gamma^*(u) - \gamma^*(u')\|_\beta \leq \frac{1}{1-\theta_\lambda(\beta)}\|u - u'\| \quad (1)$$

where $0 < \theta_\lambda(\beta) < 1$ is a uniform contraction constant depending on β . Furthermore, let $\phi_2(u)$ be the $\mathbb{E}^{\lambda_2}$ -component of $\gamma^(u)$'s initial point, then ϕ_2 is in $C^k(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ and*

$$W^{\lambda_1} = \text{graph}(\phi_2), \quad \phi_2(0) = 0, \quad \text{and} \quad D\phi_2(0) = 0.$$

We also recall that by the Variation of Parameters Formula Theorem (VPF) for splitting $\mathbb{R}^d \cong \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ corresponding to $Df(\bar{q}) \cong \text{diag}(A_1, A_2)$, the map $(\bar{x}, \bar{y}) = f(x, y)$ with $(x, y), (\bar{x}, \bar{y}) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ is equivalent to

$$\begin{cases} \bar{x} = A_1 x + h_1(x, y) \\ y = A_2^{-1} \bar{y} + h_2(\bar{x}, \bar{y}), \end{cases} \quad (2)$$

and for any orbit, $p_n = (x_n, y_n) = f(x_{n-1}, y_{n-1})$, and $n \geq 0$

$$\begin{cases} x_n = A_1^n x_0 + \sum_{i=1}^n A_1^{n-i} h_1(p_{i-1}) \\ y_n = A_2^{n-m} y_m + \sum_{i=n+1}^m A_2^{n+1-i} h_2(p_i). \end{cases} \quad (3)$$

Here, the functions h_1, h_2 are defined by f and are as smooth as f , satisfying

$$h_1(0) = 0, \quad Dh_1(0) = 0, \quad h_2(0) = 0, \quad Dh_2(0) = 0. \quad (4)$$

They are globally Lipschitz and their Lipschitz constant can be taken to be

$$L = \|D(h_1, h_2)\|_0 \rightarrow 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \rightarrow 0. \quad (5)$$

The result above holds for sufficiently small $\|f - Df(\bar{q})\|_1$.

Associated with h_i , we will need the following functions throughout

$$g_i(q, \delta p) = h_i(q + \delta p) - h_i(q), \quad \text{for } i = 1, 2. \quad (6)$$

Because $h_i \in C^{k,1}$ so is $g_i \in C^{k,1}$ satisfying

$$g_i(0,0) = 0, D_p g_i(0,0) = 0, D_{\delta p} g_i(0,0) = 0, \text{ for } i = 1, 2. \quad (7)$$

More importantly, all derivatives in q satisfy

$$D_q^j g_i(q,0) = 0, \text{ for } 0 \leq j \leq k, \text{ and } i = 1, 2. \quad (8)$$

To save notation, we will use the same notation for Lipschitz constants of both h_i and g_i

$$L = \max\{\|D(h_1, h_2)\|_0, \|D(g_1, g_2)\|_0\} \rightarrow 0 \text{ as } \|f - Df(\bar{q})\|_1 \rightarrow 0. \quad (9)$$

Since $g_i \in C^{k,1}$, we will denote by $L_1, L_2, \dots, L_k$ the Lipschitz constants for $D_q g_i, D_q^2 g_i, \dots, D_q^k g_i$, respectively. Together with the fact that $D_q^j g_i(q,0) = 0$ we have

$$\|D_q^j g_i(q, \delta p)\| \leq L_j \|\delta p\| \text{ for } 0 \leq j \leq k, \text{ and } i = 1, 2. \quad (10)$$

Unlike L which can be made as small as possible by making $\|f - Df(\bar{q})\|_1$ small, these constants L_j are not necessarily small.

We will repeatedly use this formula for geometric sequences

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \text{ for } r \neq 1$$

and its differentiation formulas in r . We will denote throughout

$$\gamma_p = \{p_n = f^n(p)\}_{n=0}^\infty$$

the orbit of f with the initial point p , for which $p_0 = p$. The proof now consists of a sequence of lemmas below.

Lemma 1. *For any parameter α satisfying $\mu_1 < \alpha < \mu_2$, let*

$$\Delta S_\alpha := \{\delta\gamma = \{\delta p_n\}_{n=0}^\infty : \delta p_n \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}, \sup\{\alpha^{-n} \|\delta p_n\| : n \geq 0\} < \infty\} \quad (11)$$

with norm

$$\|\delta\gamma\|_\alpha = \sup\{\alpha^{-n} \|\delta p_n\| : n \geq 0\}.$$

For any $q = q_0 = (u, w) \in W^{\lambda_1}$ with $\gamma_q = \{q_n\}$ and $\delta p = \{\delta p_n\} \in \Delta S_\alpha$, let $\overline{\delta\gamma} = T(\delta\gamma)$ be defined by the equations below

$$\begin{cases} \overline{\delta x}_n = A_1^n \delta x_0 + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^\infty A_2^{n+1-i} g_2(q_i, \delta p_i) \end{cases} \quad (12)$$

Then $\overline{\delta\gamma} \in \Delta S_\alpha$. Specifically, let ν, η be any parameters satisfying

$$\mu_1 < \nu < \alpha < 1/\eta < \mu_2, \quad (13)$$

then an adapted norm can be chosen so that

$$\|\overline{\delta\gamma}\|_\alpha \leq \|\delta x_0\| + \frac{L\|\delta\gamma\|_\alpha}{\alpha-\nu} + \frac{\alpha L\|\delta\gamma\|_\alpha}{1-\alpha\eta}. \quad (14)$$

More importantly, $p \in \mathcal{F}^\mu(q)$ iff the orbit difference $\delta\gamma = \gamma_p - \gamma_q$ is a fixed point of T , i.e., $p = q + \delta p$ with $\delta p = (\delta x_0, \delta y_0)$ the initial point of $\delta\gamma$, and specifically,

$$p = \pi_\mu(u, \phi_2(u)) + (\delta x_0, \sum_{i=1}^\infty A_2^{1-i} g_2(q_i, \delta p_i)). \quad (15)$$

Proof. We first show that T is well-defined together with the bound estimate. We begin by fixing an adapted norm for the condition (13) so that the following inequalities hold

$$\|A_1\| < \nu < \alpha < 1/\eta < \|A_2\| \text{ and } \|A_2^{-1}\| < \eta < \frac{1}{\alpha}. \quad (16)$$

We now demonstrate $\overline{\delta\gamma} = \{(\overline{\delta x_n}, \overline{\delta y_n})\} \in \Delta S_\alpha$. Because $g_i(q, 0) = 0$ and $\|g_i(q, \delta p)\| \leq L\|\delta p\|$ from (8, 10) we have for $\overline{\delta x_n}$

$$\begin{aligned} \|\overline{\delta x_n}\| &\leq \|A_1^n\|\|\delta x_0\| + \sum_{i=1}^n \|A_1^{n-i}g_1(q_{i-1}, \delta p_{i-1})\| \\ &\leq \nu^n\|\delta x_0\| + \sum_{i=1}^n \nu^{n-i}L\alpha^{i-1}\|\delta\gamma\|_\alpha \\ &= \nu^n\|\delta x_0\| + L\|\delta\gamma\|_\alpha \frac{\alpha^n - \nu^n}{\alpha - \nu} \\ &\leq (\|\delta x_0\| + \frac{L\|\delta\gamma\|_\alpha}{\alpha - \nu})\alpha^n. \end{aligned} \quad (17)$$

Similarly,

$$\begin{aligned} \|\overline{\delta y_n}\| &\leq \sum_{i=n+1}^\infty \|A_2^{n+1-i}g_2(q_i, \delta p_i)\| \\ &\leq \sum_{i=n+1}^\infty \eta^{i-n-1}L\alpha^i\|\delta\gamma\|_\alpha \\ &= \eta^{-n-1}L\|\delta\gamma\|_\alpha \frac{(\alpha\eta)^{n+1}}{1 - \alpha\eta} \\ &= \frac{\alpha L\|\delta\gamma\|_\alpha}{1 - \alpha\eta}\alpha^n. \end{aligned} \quad (18)$$

Hence, the estimate (14) holds. This shows that the infinite series converges uniformly and that T is well-defined, mapping ΔS_α into itself.

Next, we show the last part of the lemma. First, for $p \in \mathcal{F}^\mu(q)$, both orbits γ_p, γ_q are in S_β , and the orbit difference

$$\delta\gamma = \gamma_p - \gamma_q = \{\delta p_n : \delta p_n = p_n - q_n, n \geq 0\} \quad (19)$$

is in ΔS_α by definition. By the VPF (3), $\delta\gamma$ satisfies

$$\begin{cases} \delta x_n = A_1^n \delta x_0 + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1}) \\ \delta y_n = A_2^{n-m} \delta y_m + \sum_{i=n+1}^m A_2^{n+1-i} g_2(q_i, \delta p_i) \end{cases}$$

Because $\|\delta y_m\| \leq \alpha^m \|\delta\gamma\|_\alpha$ and $\|A_2^{n-m}\| \leq \eta^{m-n}$ and $\alpha\eta < 1$, the first term in the y_n -equation above converges to 0 as $m \rightarrow \infty$. The estimate (18) also shows the partial sum of the y_n -equation converges uniformly. Therefore the limit as $m \rightarrow \infty$ exists for the y_n -equation and the limit is exactly the y_n -equation for the map T . Hence, $\delta\gamma$ is a fixed point of T .

Conversely, assume $\delta\gamma = \{(\delta x_n, \delta y_n)\}$ is a fixed point of T for a given γ_q from W^{λ_1} . It is straightforward to verify

$$\begin{cases} \delta x_n = A_1 \delta x_{n-1} + g_1(q_{n-1}, \delta p_{n-1}) \\ \delta y_n = A_2^{-1} \delta y_{n+1} + g_2(q_{n+1}, \delta p_{n+1}). \end{cases}$$

Denote $p_n = q_n + \delta p_n$, $p_n = (x_n, y_n)$, $q_n = \pi_\mu(q_n) = (x_{q,n}, y_{q,n}) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$. Then because γ_q is an orbit it satisfies

$$\begin{cases} x_{q,n} = A_1 x_{q,n-1} + h_1(q_{n-1}) \\ y_{q,n} = A_2^{-1} y_{q,n+1} + h_2(q_{n+1}). \end{cases}$$

Sum up these two equations component by component to obtain

$$\begin{cases} x_n = A_1 x_{n-1} + h_1(p_{n-1}) \\ y_n = A_2^{-1} y_{n+1} + h_2(p_{n+1}), \end{cases}$$

which shows $\gamma_p = \{p_n\} = \gamma_q + \delta\gamma$ must be an orbit of f . Since $\gamma_q \in S_\beta$ and $\delta\gamma \in \Delta S_\alpha \subset S_\beta$, we also have $\gamma_p \in S_\beta$. Hence, the initial point, p_0 , of γ_p is in W^{λ_1} and in $\mathcal{F}^\mu(q)$ by definition. Last, the identity (15) follows by writing out the initial point of γ_p because $q = (u, \phi_2(u)) \in W^{\lambda_1}$. $\square$

Lemma 2. *Let $\phi_2 \in C^1(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ be the function whose graph is W^{λ_1} . Then there is a function $\psi_2 : \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \rightarrow \mathbb{E}^{\mu_2}$ so that for all $u \in \mathbb{E}^{\lambda_1}$,*

$$\phi_2(u) = \pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u))$$

and for every $q \in W^{\lambda_1}$ with $q = (u, \phi_2(u))$

$$\mathcal{F}^\mu(u) := \mathcal{F}^\mu(q) = \text{graph}(\psi_2(u, \cdot)). \quad (20)$$

Moreover, the definition of $\mathcal{F}^\mu$ is independent of any two different choices in α .

Proof. By Lemma 1, $p \in \mathcal{F}^\mu(q)$ iff $p = q + \delta p$ with $\delta p = \delta p_0$ the initial point of a fixed point $\delta\gamma = \{\delta p_n\}_{n \geq 0}$ of the map T from its proof. We already know from Proposition 1 that q is parameterized by $u \in \mathbb{E}^{\lambda_1}$ by $q = (u, \phi_2(u))$ as well as its orbit $\gamma_q = \gamma^*(u) = \{q_n(u)\}$. We only need to show $\delta p = (\delta x_0, \delta y_0)$ exists and is parameterized by u and by its $\mathbb{E}^{\mu_1}$ -coordinate δx_0 which we will replace by $\delta x = \delta x_0 \in \mathbb{E}^{\mu_1}$. In fact, if that is true, then in this parameterized designation, the function ψ_2 must be defined from the identity (15) as below

$$p = (x, y) = (x, \psi_2(u, x)) := \pi_\mu(u, \phi_2(u)) + (\delta x, \delta y_0(u, \delta x))$$

where $\delta x = x - \pi_{\mu_1}(q) = x - \pi_{\mu_1}(u)$, namely

$$\psi_2(u, x) = \pi_{\mu_2}(u, \phi_2(u)) + \sum_{i=1}^{\infty} A_2^{1-i} g_2(q_i(u), \delta p_i(u, x - \pi_{\mu_1}(u))). \quad (21)$$

Assuming the fixed point $\delta\gamma$ is unique for T , then we see the zero sequence $\delta\gamma = \{0\}$ is a trivial fixed point if $\delta x = 0$. As a consequence, we get

$$\psi_2(u, \pi_{\mu_1}(u)) = \pi_{\mu_2}(u, \phi_2(u)) + \delta y_0(u, 0) = \pi_{\mu_2}(u, \phi_2(u))$$

which gives

$$\pi_{\lambda_2}(\psi_2)(u, \pi_{\mu_1}(u)) = \phi_2(u),$$

the inclusion of W^{λ_1} . Definition (21) obviously shows (20). Therefore, it is only left to show the existence and uniqueness of fixed point of T for each u , δx , and their independence on any two choices in α .

To this end, we will consider T as a parameterized map $T : \Delta S_\alpha \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1} \rightarrow \Delta S_\alpha$ with $\overline{\delta\gamma} = T(\delta\gamma, u, \delta x)$ being defined by (12) as below

$$\begin{cases} \overline{\delta x}_n = A_1^n \delta x + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}(u), \delta p_{i-1}) \\ \overline{\delta y}_n = \sum_{i=n+1}^{\infty} A_2^{n+1-i} g_2(q_i(u), \delta p_i) \end{cases} \quad (22)$$

We first show T is a uniform contraction. By the proof of Lemma 1, $T(\cdot, u, \delta x)$ maps ΔS_α into ΔS_α . For its uniform contraction, let $\delta\gamma, \delta\gamma' \in \Delta S_\alpha$ and $\overline{\delta\gamma} = T(\delta\gamma, u, \delta x)$, $\overline{\delta\gamma'} = T(\delta\gamma', u, \delta x)$. Then we have

$$\begin{aligned} \|\overline{\delta x_n} - \overline{\delta x_n'}\| &\leq \sum_{i=1}^n \|A_1^{n-i}[g_1(q_{i-1}(u), \delta p_{i-1}) - g_1(q_{i-1}(u), \delta p'_{i-1})]\| \\ &\leq \sum_{i=1}^n \nu^{n-i} L \|\delta p_{i-1} - \delta p'_{i-1}\| \\ &\leq \sum_{i=1}^n \nu^{n-i} L \alpha^{i-1} \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \frac{L}{\alpha - \nu} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha \end{aligned} \quad (23)$$

and

$$\begin{aligned} \|\overline{\delta y_n} - \overline{\delta y_n'}\| &\leq \sum_{i=n+1}^\infty \|A_2^{n+1-i}[g_2(q_i(u), \delta p_i) - g_2(q_i(u), \delta p'_i)]\| \\ &\leq \sum_{i=n+1}^\infty \eta^{i-n-1} L \|\delta p_i - \delta p'_i\| \\ &\leq \sum_{i=n+1}^\infty \eta^{i-n-1} \alpha^i \|\delta\gamma - \delta\gamma'\|_\alpha \\ &\leq \frac{L\alpha}{1 - \alpha\eta} \alpha^n \|\delta\gamma - \delta\gamma'\|_\alpha. \end{aligned} \quad (24)$$

Hence,

$$\|T(\delta\gamma, u, \delta x) - T(\delta\gamma', u, \delta x)\|_\alpha \leq \left(\frac{L}{\alpha - \nu} + \frac{L\alpha}{1 - \alpha\eta}\right) \|\delta\gamma - \delta\gamma'\|_\alpha$$

showing $T(\cdot, u, \delta x)$ is a uniform contraction in ΔS_α provided

$$\theta := \theta(\alpha) = \frac{L}{\alpha - \nu} + \frac{L\alpha}{1 - \alpha\eta} < 1 \quad (25)$$

which is true for sufficiently small $\|f - Df(\bar{q})\|_1$. We denote the fixed point by

$$\delta\gamma^*(u, \delta x) = \{\delta p_n(u, \delta x) = (\delta x_n(u, \delta x), \delta y_n(u, \delta x))\}_{n=0}^\infty \quad (26)$$

Notice that the existence and uniqueness proof of $\delta\gamma^*$ above shows that for any

$$\mu_1 < \alpha' < \alpha < \mu_2 \quad \text{with } \|A_1\| < \nu < \alpha' < \alpha < 1/\eta < \|A_2\|, \|A_2^{-1}\| < \eta$$

as long as

$$\theta(\alpha'), \theta(\alpha) < 1$$

$T(\cdot, u, \delta x)$ has a unique fixed point in $\Delta S_{\alpha'}$ and ΔS_α . But since $\Delta S_{\alpha'}$ is a closed subspace of ΔS_α , the unique fixed point $\delta\gamma^*(u, \delta x)$ is in both $\Delta S_{\alpha'}$ and ΔS_α . This shows the independence of $\mathcal{F}^\mu$ on any two choices in α . $\square$

Lemma 3. *The foliation function ψ_2 is $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \mathbb{E}^{\mu_2})$.*

Proof. Notice from its definition (21) that we only need to show δp_0 is C^1 for which it suffices to show the unique fixed point $\delta\gamma^*$ of T from Lemma 2 is C^1 since δp_0 is only a point of the sequence $\delta\gamma^*$. By the Uniform Contraction Principle II, we need to verify two conditions: (1) $T(\delta\gamma, u, \delta x)$ is differentiable in $\delta\gamma$ and $\|D_{\delta\gamma} T(\delta\gamma, u, \delta x)\|$ is uniformly bounded by a constant smaller than 1; (2) $T \in C^1(\Delta S_\alpha \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$.

To show (1), let $\delta\gamma = \{\delta p_n\}, v = \{v_n\} \in \Delta S_\alpha$, and formally differentiate (22). Then $D_{\delta\gamma}T(\delta\gamma, u, \delta x)v$ needs to be as below in components:

$$\begin{cases} [D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,1} = \sum_{i=1}^n A_1^{n-i} D_{\delta p} g_1(q_{i-1}(u), \delta p_{i-1}) v_{i-1} \\ [D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_{\delta p} g_2(q_i(u), \delta p_i) v_i. \end{cases} \quad (27)$$

By the exactly same estimate as for (23) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,1}\| \leq \frac{L}{\alpha-\nu} \alpha^n \|v\|_\alpha.$$

Similarly, by the exactly same estimate as for (24) we have

$$\|[D_{\delta\gamma}T(\delta\gamma, u, \delta x)v]_{n,2}\| \leq \frac{L\alpha}{1-\alpha\eta} \alpha^n \|v\|_\alpha.$$

These estimates imply two conclusions. One, because of the uniform convergence of the second equation, the derivative $D_{\delta\gamma}T(\delta\gamma, \delta x)$ exists. Two, it shows the derivative is a bounded linear map in $L(\Delta S_\alpha, \Delta S_\alpha)$ whose α -norm

$$\|D_{\delta\gamma}T(\delta\gamma, u, \delta x)\|_\alpha \leq \theta(\alpha) < 1,$$

is bounded by the same uniform contraction constant $\theta(\alpha)$ from (25).

To show (2), we separate it into three cases. The first case is done above for derivative in $\delta\gamma$, the second case is for derivative in δx , and the third case is for derivative in u . For the second case we have formally

$$[D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,1} = A_1^n, \text{ and } [D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,2} = 0,$$

which implies

$$\|[D_{\delta x}T(\delta\gamma, u, \delta x)]_n\| \leq \|A_1^n\| \leq \alpha^n$$

and $\|D_{\delta x}T(\delta\gamma, u, \delta x)\|_\alpha \leq 1$ follows. That is, T is C^1 in δx .

To show the third case, we will treat T in u as a composition

$$T(\delta\gamma, u, \delta x) = \bar{T}(\delta\gamma, \gamma^*(u), \delta x) \quad (28)$$

of a map $\bar{T} : \Delta S_\alpha \times S_\beta \times \mathbb{E}^{\mu_1} \rightarrow \Delta S_\alpha$ with a C^1 map $\gamma^* : \mathbb{E}^{\lambda_1} \rightarrow S_\beta$ which is the orbit sequence $\gamma^*(u) = \gamma_q$ for point $q = (u, w) \in W^{\lambda_1}$. Here, $\bar{\delta\gamma} = \bar{T}(\delta\gamma, \gamma, \delta x)$ is defined as below

$$\begin{cases} \bar{\delta x}_n = A_1^n \delta x + \sum_{i=1}^n A_1^{n-i} g_1(q_{i-1}, \delta p_{i-1}) \\ \bar{\delta y}_n = \sum_{i=n+1}^\infty A_2^{n+1-i} g_2(q_i, \delta p_i), \end{cases} \quad (29)$$

the same definition as T except for a general $\gamma = \{q_n\}_{n=0}^\infty \in S_\beta$. Because of the composition we only need to show $\bar{T}$ is C^1 in γ .

To this end, by the sub-tight split definition that $\mu_1 \lambda_1 < \mu_2$, we can choose a parameter ς close to μ_1 , α close to μ_2 , and β close to λ_1 so that

$$\varsigma\beta < \alpha \text{ where } \mu_1 < \varsigma < \alpha < \mu_2 \leq \lambda_1 < \beta < \lambda_2, \quad (30)$$

for which an adjusted norm can be chosen to satisfy (16) and

$$\|A_1\| < \nu < \varsigma < \alpha < \|A_2\|. \quad (31)$$

We will treat the fixed point $\delta\gamma^*(u, \delta x)$ in both ΔS_ς and ΔS_α . We also use the estimates below from (10) that for $i = 1, 2$,

$$\|g_i(q, \delta p) - g_i(q', \delta p)\| \leq \|D_q g_i(\cdot, \delta p)\|_0 \|q - q'\| \leq L_1 \|\delta p\| \|q - q'\|. \quad (32)$$

Now, formally differentiating $\bar{T}$ in γ , we obtain from (29) any $v \in S_\beta$

$$\begin{cases} [D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,1} = \sum_{i=1}^n A_1^{n-i} D_q g_1(q_{i-1}, \delta p_{i-1}) v_{i-1} \\ [D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_q g_2(q_i, \delta p_i) v_i, \end{cases} \quad (33)$$

Because of $\varsigma\beta < \alpha$, we have

$$\begin{aligned} \|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,1}\| &\leq \sum_{i=1}^n \|A_1^{n-i}\| L_1 \|\delta p_{i-1}\| \|v_{i-1}\| \\ &\leq L_1 \sum_{i=1}^n \nu^{n-i} \varsigma^{i-1} \|\delta\gamma\|_\varsigma \beta^{i-1} \|v\|_\beta \\ &\leq L_1 \|\delta\gamma\|_\alpha \sum_{i=1}^n \nu^{n-i} \alpha^{i-1} \|v\|_\beta \\ &\leq \frac{L_1 \|\delta\gamma\|_\alpha}{\alpha - \nu} \alpha^n \|v\|_\beta. \end{aligned} \quad (34)$$

Similarly, we have

$$\begin{aligned} \|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,2}\| &\leq \sum_{i=n+1}^\infty \|A_2^{n+1-i}\| L_1 \|\delta p_i\| \|v_i\| \\ &\leq L_1 \sum_{i=n+1}^\infty \eta^{i-n-1} \varsigma^i \|\delta\gamma\|_\varsigma \beta^i \|v\|_\beta \\ &\leq L_1 \|\delta\gamma\|_\alpha \sum_{i=n+1}^\infty \eta^{i-n-1} \alpha^i \|v\|_\beta \\ &\leq \frac{L_1 \|\delta\gamma\|_\alpha \alpha}{1 - \alpha\eta} \alpha^n \|v\|_\beta. \end{aligned} \quad (35)$$

Combine these two estimates to obtain

$$\|[D_\gamma \bar{T}(\delta\gamma, \gamma, \delta x)]\|_\alpha \leq \left(\frac{1}{\alpha - \nu} + \frac{\alpha}{1 - \alpha\eta}\right) L_1 \|\delta\gamma\|_\alpha.$$

The convergence of the infinite series also shows the derivative exists. Hence $\bar{T}(\delta\gamma, \cdot, \delta x)$ is in $C^1(S_\beta, \Delta S_\alpha)$ and T is C^1 in u as needed. $\square$

Lemma 4. *f is Lipschitz on $\mathcal{F}^\mu(q)$ and for the adapted norm from Lemma 1 the Lipschitz constant is $\leq \alpha$ uniformly for all $q \in W^{\lambda_1}$.*

Proof. We remark first that since $T(\delta\gamma, u, \delta x)$ is Lipschitz continuous in δx with

$$\|T(\delta\gamma, u, \delta x) - T(\delta\gamma, u, \delta x')\|_\alpha \leq \|\delta x - \delta x'\|,$$

because $\|A_1^n\| < \nu^n$, we have by the Uniform Contraction Principle I that $\delta\gamma^*$ satisfies

$$\|\delta\gamma^*(u, \delta x) - \delta\gamma^*(u, \delta x')\|_\alpha \leq \frac{1}{1 - \theta} \|\delta x - \delta x'\|. \quad (36)$$

For the proof of the lemma, we need to show that for any $q \in W^{\lambda_1}$ and for any $p, p' \in \mathcal{F}^\mu(q)$, $\|f(p) - f(p')\| \leq \alpha \|p - p'\|$. Let $\gamma_p, \gamma_{p'}$ be the orbits through p, p' , respectively. Then $\delta\gamma^* = \gamma_p - \gamma_q$ and $\delta\gamma^{*'} = \gamma_{p'} - \gamma_q$ are fixed points of

$T(\cdot, u, \delta x)$ and $T(\cdot, u, \delta x')$, respectively, with $\delta x = x - \pi_{\mu_1}(u)$, $\delta x' = x' - \pi_{\mu_1}(u)$, $\delta\gamma^* = \delta\gamma^*(u, \delta x)$, and $\delta\gamma^{*'} = \delta\gamma^*(u, \delta x')$. More importantly,

$$\gamma_p - \gamma_{p'} = (\gamma_p - \gamma_q) - (\gamma_{p'} - \gamma_q) = \delta\gamma^* - \delta\gamma^{*'}$$

whose second point on the sequence is

$$f(p) - f(p') = p_1 - p'_1 = \delta p_1(u, \delta x) - \delta p_1(u, \delta x').$$

The $\mathbb{E}^{\mu_1}$ -coordinate of the right side can be estimated as

$$\begin{aligned} \|\overline{\delta x_1} - \overline{\delta x_1}'\| &\leq \|A_1(\delta x - \delta x') + g_1(q_0(u), \delta x) - g_1(q_0(u), \delta x')\| \\ &\leq \nu \|\delta x - \delta x'\| + \|h_1(q_0(u) + \delta x) - h_1(q_0(u) + \delta x')\| \\ &\leq \nu \|\delta x - \delta x'\| + L \|\delta x - \delta x'\| \\ &\leq (\nu + L) \|p - p'\| \end{aligned}$$

The $\mathbb{E}^{\mu_2}$ -coordinate of the right side is

$$\begin{aligned} \|\overline{\delta y_1} - \overline{\delta y_1}'\| &\leq \sum_{i=2}^{\infty} \|A_2^{2-i} [g_2(q_i(u), \delta p_i) - g_2(q_i(u), \delta p'_i)]\| \\ &= \sum_{i=2}^{\infty} \|A_2^{2-i} [h_2(q_i(u) + \delta p_i) - h_2(q_i(u) + \delta p'_i)]\| \\ &\leq \sum_{i=2}^{\infty} \eta^{i-2} L \alpha^i \|\delta\gamma^* - \delta\gamma^{*'}\|_{\alpha} \\ &\leq \frac{L\alpha^2}{1-\alpha\eta} \|\delta\gamma^*(u, \delta x) - \delta\gamma^*(u, \delta x')\|_{\alpha} \\ &\leq \frac{L\alpha^2}{1-\alpha\eta} \frac{1}{1-\theta} \|\delta x - \delta x'\| \\ &\leq \frac{L\alpha^2}{1-\alpha\eta} \frac{1}{1-\theta} \|p - p'\| \end{aligned}$$

where (36) is used for the second last estimate. Therefore,

$$\|f(p) - f(p')\| \leq (\nu + L + \frac{L\alpha^2}{1-\alpha\eta} \frac{1}{1-\theta}) \|p - p'\| \leq \alpha \|p - p'\|$$

for sufficiently small L , i.e., for sufficiently small $\|f - Df(\bar{q})\|_1$ since $\nu < \alpha$. $\square$

Lemma 5. *If f is $C^{k,1}$, $\lambda_1^k < \lambda_2$, $\mu_1^k < \mu_2$, and $\mu_1 \lambda_1^k < \mu_2$, then ψ_2 is C^k .*

Proof. The case of $k = 1$ is done in Lemma 3. For the case of $k \geq 2$, we notice first from the definition of ψ_2 , (21), that we need to prove both ϕ_2 and $\delta p_0(u, \delta x)$ are C^k . By the λ -left manifold theorem we know from Proposition 1 that ϕ_2 is C^k because $\lambda_1^k < \lambda_2$. Hence, we only need to show $\delta p_0(u, \delta x)$ is C^k , which in turn suffices to show the fixed point function $\delta\gamma^*(u, \delta x)$ is C^k . Similar to the proof for the λ -left manifold theorem, the Uniform Contraction Principle II cannot be applied directly for $k = 2$ (except for the special case where $\mu_1 \leq 1$). This is because we cannot prove in general $T \in C^k(\Delta S_{\alpha} \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_{\alpha})$. An indirect approach is needed.

First, we want to prove instead the following claim

$$T \in C^k(\Delta S_{\varsigma} \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_{\alpha}) \quad (37)$$

where ς is sufficiently close to μ_1 and α is sufficiently close to μ_2 satisfying

$$\mu_1 < \varsigma < \alpha < \mu_2 \quad \text{and} \quad \mu_1^k < \varsigma^k < \alpha < \mu_2 \quad (38)$$

which is guaranteed by the assumption $\mu_1^k < \mu_2$.

To prove the claim, we first fix an adapted norm for $(x, y) \in \mathbb{E}^{\mu_1} \times \mathbb{E}^{\mu_2}$ so that in addition to (16) we also have

$$\|A_1\| < \nu < \varsigma < \alpha < \|A_2\|. \quad (39)$$

We separate the proof into four cases. The first case is for derivatives in δx , the second case is for derivatives in $\delta\gamma$, the third case is for derivatives in u , and the fourth case is about mixed derivatives. For the first case we already have from the proof of Lemma 3 that

$$[D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,1} = A_1^n, \quad \text{and} \quad [D_{\delta x}T(\delta\gamma, u, \delta x)]_{n,2} = 0,$$

i.e., T is C^1 in δx . And $D_{\delta x}^j T(\delta\gamma, u, \delta x) = 0$, for $2 \leq j \leq k$, the zero operators. Hence, it is obvious that $T(\delta\gamma, u, \cdot) \in C^k(\mathbb{E}^{\mu_1}, \Delta S_\alpha)$ for $\delta\gamma \in \Delta S_\varsigma$.

For the second case, applying the case of $j = 1$ for $\mu_1 < \varsigma < \alpha < \mu_2$, we know $D_{\delta\gamma}T(\delta\gamma, u, \delta x) \in L(\Delta S_\varsigma, \Delta S_\varsigma) \subset L(\Delta S_\varsigma, \Delta S_\alpha)$ because $\Delta S_\varsigma \subset \Delta S_\alpha$ holds automatically. For any $2 \leq j \leq k$, $[D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)]$ needs to be a j -linear form ΔS_ς to ΔS_α . To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in \Delta S_\varsigma$, $1 \leq \ell \leq j$. Formally differentiate (22) to get

$$\begin{cases} [D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)v]_{n,1} = \sum_{i=1}^n A_1^{n-i} D_{\delta p}^j g_1(q_{i-1}(u), \delta p_{i-1}) v_{i-1} \\ [D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)v]_{n,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_{\delta p}^j g_2(q_i(u), \delta p_i) v_i, \end{cases} \quad (40)$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (23) and because $g_i(q, \delta p) = h_i(q + \delta p) - h_i(q)$, $\varsigma < \alpha$, and $\varsigma^k < \alpha$, we have

$$\begin{aligned} \|[D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)v]_{n,1}\| &\leq \sum_{i=1}^n \|A_1^{n-i}\| \|D^j h_1\| \|v_{i-1}\| \\ &\leq \sum_{i=1}^n \nu^{n-i} \|h_1\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \|h_1\|_k \sum_{i=1}^n \nu^{n-i} \varsigma^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \|h_1\|_k \sum_{i=1}^n \nu^{n-i} \alpha^{(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \frac{\|h_1\|_k \alpha}{\alpha - \nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \quad (41)$$

Here we used the property that $\varsigma < \alpha$ and $\varsigma^k < \alpha$ imply $\varsigma^j < \alpha$ for $1 \leq j \leq k$. Similar to (24) we have

$$\begin{aligned} \|[D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)v]_{n,2}\| &\leq \sum_{i=n+1}^\infty \|A_2^{n+1-i}\| \|D^j h_2\| \|v_i\| \\ &\leq \sum_{i=n+1}^\infty \eta^{i-n-1} \|h_2\|_j \varsigma^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \|h_2\|_k \eta^{-n-1} \sum_{i=n+1}^\infty (\eta\alpha)^i \Pi_{\ell=1}^j \|v^\ell\|_\alpha \\ &\leq \frac{\|h_2\|_k \alpha}{1 - \alpha\eta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\alpha. \end{aligned} \quad (42)$$

Combine these two estimates to obtain

$$\|[D_{\delta\gamma}^j T(\delta\gamma, u, \delta x)]\|_\alpha \leq \|(h_1, h_2)\|_k \max\left\{\frac{1}{\alpha - \nu}, \frac{\alpha}{1 - \alpha\eta}\right\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that $T(\cdot, u, \delta x) \in C^k(\Delta S_\varsigma, \Delta S_\alpha)$.

For the third case $T(\delta\gamma, \cdot, \delta x) \in C^k(\mathbb{E}^{\lambda_1}, \Delta S_\alpha)$, we will treat it similarly as in the proof of Lemma 3 to be a composition of $\bar{T}(\delta\gamma, \cdot, \delta x) : S_\beta \rightarrow \Delta S_\alpha$, defined by (28, 29), with $\gamma = \gamma^* : \mathbb{E}^{\lambda_1} \rightarrow S_\beta$ for each $\delta\gamma \in \Delta S_\varsigma$. By Proposition 1, $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$. So we only need to show $\bar{T}(\delta\gamma, \cdot, \delta x)$ is C^k . The $k = 1$ case is treated in the proof of Lemma 3.

For $k \geq 2$, we use the condition that $\mu_1 \lambda_1^k < \mu_2$ to fix parameter ς close to μ_1 , β close to λ_1 , and α close to μ_2 so that in addition to (38) we also have

$$\varsigma \beta^k < \alpha \quad \text{where} \quad \mu_1 < \varsigma < \alpha < \mu_2 \leq \lambda_1 < \beta < \lambda_2. \quad (43)$$

As a result we assume that an adapted norm is chosen so that both (16) and (39) hold. We will also use the property (10) that

$$\|D_q^j g_i(q, \delta p)\| \leq \bar{L} \|\delta p\| \quad \text{with} \quad \bar{L} = \max\{L_j : 1 \leq j \leq k\}. \quad (44)$$

We are now ready to show $\bar{T}(\delta\gamma, \cdot, \delta x) \in C^k(S_\beta, \Delta S_\alpha)$ for $\delta\gamma \in \Delta S_\varsigma$ $\delta x \in \mathbb{E}^{\mu_1}$. We need to show $[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)]$ is a bounded j -linear form from S_β to ΔS_α . To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\beta$, $1 \leq \ell \leq j$. Formally differentiate (29) in $\gamma \in S_\beta$ to get

$$\begin{cases} [D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,1} = \sum_{i=1}^n A_1^{n-i} D_q^j g_1(q_{i-1}, \delta p_{i-1}) v_{i-1} \\ [D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,2} = \sum_{i=n+1}^\infty A_2^{n+1-i} D_q^j g_2(q_i, \delta p_i) v_i, \end{cases} \quad (45)$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (34) and because of (44), we have

$$\begin{aligned} \|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,1}\| &\leq \sum_{i=1}^n \|A_1^{n-i}\| L_j \|\delta p_{i-1}\| \|v_{i-1}\| \\ &\leq \bar{L} \sum_{i=1}^n \nu^{n-i} \varsigma^{i-1} \|\delta\gamma\|_\varsigma \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \bar{L} \|\delta\gamma\|_\alpha \sum_{i=1}^n \nu^{n-i} \varsigma^{i-1} \beta^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \bar{L} \|\delta\gamma\|_\alpha \sum_{i=1}^n \nu^{n-i} \alpha^{i-1} \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \frac{\bar{L} \|\delta\gamma\|_\alpha}{\alpha - \nu} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\beta, \end{aligned}$$

where $\varsigma \beta^j < \alpha$, $1 \leq j \leq k$ because $\varsigma < \alpha$ and $\varsigma \beta^k < \alpha$. Similar to the estimate of (35) we have

$$\begin{aligned} \|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)v]_{n,2}\| &\leq \sum_{i=n+1}^\infty \|A_2^{n+1-i}\| L_j \|\delta p_i\| \|v_i\| \\ &\leq \bar{L} \sum_{i=n+1}^\infty \eta^{i-n-1} \varsigma^i \|\delta\gamma\|_\varsigma \beta^{ji} \Pi_{\ell=1}^j \|v_i^\ell\|_\beta \\ &\leq \bar{L} \|\delta\gamma\|_\alpha \sum_{i=n+1}^\infty \eta^{i-n-1} \alpha^i \Pi_{\ell=1}^j \|v^\ell\|_\beta \\ &\leq \frac{\bar{L} \|\delta\gamma\|_\alpha}{1 - \alpha\eta} \alpha^n \Pi_{\ell=1}^j \|v^\ell\|_\beta. \end{aligned}$$

Combine these two estimates to obtain

$$\|[D_\gamma^j \bar{T}(\delta\gamma, \gamma, \delta x)]\|_\alpha \leq \left(\frac{1}{\alpha - \nu} + \frac{\alpha}{1 - \alpha\eta}\right) \bar{L} \|\delta\gamma\|_\alpha.$$

The convergence of the infinite series also shows the derivatives are well-defined. Hence $\bar{T}(\delta\gamma, \cdot, \delta x)$ is in $C^k(S_\beta, \Delta S_\alpha)$, showing T is C^k in u .

For the fourth case about mixed derivatives of T in all variables, the arguments above for $\delta\gamma, u, \delta x$ can be combined to show all derivatives up to order k exist for T . This completes the proof for the claim (37).

We are now ready to show $\delta\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. By the Uniform Contraction Principle II for $T \in C^1(\Delta S_\zeta \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\zeta)$, the fixed point $\delta\gamma^*(\cdot)$ is in $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\zeta)$ and its derivative in $(u, \delta x)$ is given by

$$D\delta\gamma^*(\cdot) = \sum_{n=0}^{\infty} [D_\gamma T(\delta\gamma^*(\cdot), \cdot)]^n D_{(u, \delta x)} T(\delta\gamma^*(\cdot), \cdot).$$

Since $\delta\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\zeta)$, $T \in C^1(\Delta S_\zeta \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\zeta) \subset C^1(\Delta S_\zeta \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and $T \in C^k(\Delta S_\zeta \times \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, $k \geq 2$ by the claim, here is the key to notice that the composition $D_\gamma T(\delta\gamma^*(\cdot), \cdot)$ is $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. This implies that the infinite series on the right is in $C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and therefore, $D\delta\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and $\delta\gamma^*(\cdot) \in C^2(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$ follows. Apply this argument recursively to obtain $\delta\gamma^*(\cdot) \in C^3(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$, and so on until we reach $\delta\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \Delta S_\alpha)$. As a component of the initial point of $\delta\gamma^*$, δp_0 is in $C^k(\mathbb{E}^{\lambda_1} \times \mathbb{E}^{\mu_1}, \mathbb{E}^{\mu_2})$, completing the proof that ψ_2 is C^k . $\square$

Proof of Theorem 1. After the preceding lemmas, it only remains to point out that by the definition of W^{μ_1} , which is an invariant subspace of W^{λ_1} , it coincides with the definition of the foliation through the fixed point, $\mathcal{F}^\mu(\bar{q})$, i.e., $W^{\mu_1} = \mathcal{F}^\mu(\bar{q})$. In fact, we can show the tangent space directly as below. Since $\bar{q} \sim u = 0$ and $\phi_2(0) = 0$, we have from (21)

$$\psi_2(0, x) = \delta y_0(0, x) = \sum_{i=1}^{\infty} A_2^{1-i} h_2(\delta p_i(0, x))$$

whose partial derivative in $x \in \mathbb{E}^{\mu_1}$ at the fixed point $\bar{q} \sim x = 0$ is

$$D_x \psi_2(0, 0) = \sum_{i=1}^{\infty} A_2^{1-i} Dh_2(\delta p_i(0, 0)) D_x \delta p_i(0, 0) = 0$$

since $\delta\gamma^*(0, 0) = \{0\}$ and $Dh_i(0) = 0$, showing $\mathbb{T}_{\bar{q}} \mathcal{F}^\mu(0) = \mathbb{E}^{\mu_1}$. $\square$

Remark: We can see from the proofs above that if the λ -left manifold point q is fixed at the fixed point $\bar{q}$ throughout, then the extra Lipschitz continuity condition for the highest derivative of f is not needed. That is, the μ -left manifold $\mathcal{F}^\mu(\bar{q}) = W^{\mu_1}$ is C^k if f is C^k plus $\mu_1^k < \mu_2$. This is because in this case, $g(\bar{q}, \delta p) = h(p)$ with $\bar{q} = 0$, $\delta p = p$.