[Lecture Note 4]

Left and Right Manifold Theorem

The Left-Manifold Theorem obtained in this note can be used to obtain all commonly encountered local invariant manifolds of fixed points for both diffeomorphisms and ordinary differential equations. They include: the local strong-stable manifold, $W^{\rm ss}_{\rm loc}$, the local stable manifold, $W^{\rm s}_{\rm loc}$, the local center-stable manifold, $W^{\rm cs}_{\rm loc}$, the local center-manifold, $W^{\rm c}_{\rm loc}$, the local center-unstable manifold, $W^{\rm cu}_{\rm loc}$, the local unstable manifold, $W^{\rm u}_{\rm loc}$, and the local strong-unstable manifold, $W^{\rm uu}_{\rm loc}$. All of them are as smooth as f.

Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$. Let $J = Df(\bar{q})$, and denote

$$\sigma = \{\lambda \in \mathbb{C} : \ \lambda \text{ is an eigenvalue of } J. \ \}$$

Also, $|\sigma|$ denote the set of absolute values of elements from σ.

Definition 1. *Let $0 < \lambda_1 < \lambda_2$. The interval $[\lambda_1, \lambda_2]$ is called a pseudo-hyperbolic split for J if (λ_1, λ_2) is a spectral gap in the following sense*

(i) $|\sigma| \cap (\lambda_1, \lambda_2) = \varnothing$.

(ii) $\lambda_1 = \max\{|\sigma| \cap [0, \lambda_1]\}$.

(iii) $\lambda_2 = \min\{|\sigma| \cap [\lambda_2, \infty)\}$.

Denote by $\mathbb{E}^{\lambda_1}$ the generalized eigenspace of J for eigenvalues $\sigma^1 = \{\lambda \in \sigma : |\lambda| \le \lambda_1\}$ and $\mathbb{E}^{\lambda_2}$ the generalized eigenspace of J for eigenvalues $\sigma^2 = \{\lambda \in \sigma : |\lambda| \ge \lambda_2\}$. Then $\mathbb{R}^d \cong \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$.

Definition 2. *Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$ and let $[\lambda_1, \lambda_2]$ be a pseudo-hyperbolic split of $J = Df(\bar{q})$. Let β be any constant satisfying $\lambda_1 < \beta < \lambda_2$. The left or lambda-left manifold of the fixed point $\bar{q}$ for f is*

$$W^{\lambda_1} = \{p : \{\beta^{-n}[f^n(p) - \bar{q}]\}_{n=0}^{\infty} \text{ is a bounded sequence}\}.$$

The right or lambda-right manifold is

$$W^{\lambda_2} = \{p : \{\beta^{n}[f^{-n}(p) - \bar{q}]\}_{n=0}^{\infty} \text{ is a bounded sequence}\}.$$

Theorem 1 (Left Manifold Theorem)**.** *Let $\bar{q}$ be a fixed point of a diffeomorphism f in $\mathbb{R}^d$ with a pseudo-hyperbolic split $[\lambda_1, \lambda_2]$. Let $\mathbb{R}^d \cong \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$. Then a sufficiently small $\|f - Df(\bar{q})\|_1$ implies*

(i) *W^{λ_1} is the graph of a C^1 function $\phi_2 : \mathbb{E}^{\lambda_1} \to \mathbb{E}^{\lambda_2}$*

$$W^{\lambda_1} = \operatorname{graph}(\phi_2),$$

(ii) *The tangent space of W^{λ_1} at the fixed point is the lambda-left eigenspace*

$$\mathbb{T}_{\bar{q}} W^{\lambda_1} \cong \mathbb{E}^{\lambda_1}.$$

(iii) W^{λ_1} *is independent of any two different choices in* β*.*

(iv) f *is uniform Lipschitz on* W^{λ_1} *and for an adapted norm the Lipschitz constant is* $\leq \beta$*.*

(v) If ${\lambda_1}^k < \lambda_2$ *and* $f \in C^k(\mathbb{R}^d),\ 1 \leq k < \infty$*, then* $\phi_2 \in C^k(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$*. If* ${\lambda_1}^{k+1} < \lambda_2$ *and* $f \in C^{k,1}(\mathbb{R}^d)$*, then* $\phi_2 \in C^{k,1}(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$*.*

The proof is an application of the Uniform Contraction Principle. The main idea is to construct the lambda-left manifold function ϕ_2 as part of a fixed point of a uniform contraction map. We will break it up into a few lemmas.

Before doing so, we recall a few important properties about f. We first translate $\bar{q}$ to the origin and choose a coordinate system (x, y) for the splitting $\mathbb{R}^d \cong \mathbb{E}^{\lambda_1} \times \mathbb{E}^{\lambda_2}$ for which $Df(\bar{q}) \cong \operatorname{diag}(A_1, A_2)$. By the Variation of Parameters Formula Theorem, a sufficiently small $\|f - Df(\bar{q})\|_1$ implies that the map $(\bar{x}, \bar{y}) = f(x, y)$ is equivalent to

$$\begin{cases} \bar{x} = A_1 x + h_1(x, y) \\ y = {A_2}^{-1}\bar{y} + h_2(\bar{x}, \bar{y}), \end{cases} \tag{1}$$

and for any orbit, $p_n = (x_n, y_n) = f(x_{n-1}, y_{n-1})$, and $n \geq 0$,

$$\begin{cases} x_n = {A_1}^n x_0 + \sum_{i=1}^n {A_1}^{n-i} h_1(p_{i-1}) \\ y_n = {A_2}^{n-m} y_m + \sum_{i=n+1}^m {A_2}^{n+1-i} h_2(p_i). \end{cases} \tag{2}$$

Here, the functions $h_1,\ h_2$ are defined by f and are as smooth as f, satisfying

$$h_1(0) = 0,\ Dh_1(0) = 0,\ h_2(0) = 0,\ Dh_2(0) = 0 \tag{3}$$

and they are globally Lipschitz and the Lipschitz constant can be taken to be

$$L = \|(Dh_1, Dh_2)\|_0 \to 0 \quad \text{as} \quad \|f - Df(\bar{q})\|_1 \to 0. \tag{4}$$

We will repeatedly use the formula below and and its differentiations in r

$$a + ar + ar^2 + \cdots + ar^{n-1} = \tfrac{a(1-r^n)}{1-r}, \text{ for } r \neq 1.$$

Especially, its convergence and its derivatives convergence as $n \to \infty$ for $0 < r < 1$. We will denote throughout

$$\gamma_p = \{p_n = f^n(p)\}_{n=0}^{\infty}$$

the forward orbit of f with the initial point p, for which $p_0 = p$. The proof now consists of a sequence of lemmas below.

Lemma 1. *For the parameter* β *from the definition of* W^{λ_1}*, let*

$$S_\beta := \{\gamma = \{p_n\}_{n=0}^{\infty} : p_n \in \mathbb{R}^d, \sup\{\beta^{-n}\|p_n\| : n \geq 0\} < \infty\} \tag{5}$$

with norm

$$\|\gamma\|_\beta = \sup\{\beta^{-n}\|p_n\| : n \geq 0\}.$$

For any $\gamma = \{p_n = (x_n, y_0)\} \in S_\beta$*, let* $\overline{\gamma} = T(\gamma)$ *be defined by equations*

$$\left\{ \begin{array}{l} \bar{x}_n = {A_1}^n x_0 + \sum_{i=1}^n {A_1}^{n-i} h_1(p_{i-1}) \\ \bar{y}_n = \sum_{i=n+1}^\infty {A_2}^{n+1-i} h_2(p_i). \end{array} \right. \tag{6}$$

Then $\overline{\gamma} \in S_\beta$*. Specifically, let* α, ν *be parameters satisfying*

$$\lambda_1 < \nu < \beta < 1/\alpha < \lambda_2, \tag{7}$$

then an adapted norm can be chosen so that

$$\|\overline{\gamma}\|_\beta \leq \|x_0\| + \tfrac{L\|\gamma\|_\beta}{\beta - \nu} + \tfrac{L\beta\|\gamma\|_\beta}{1-\alpha\beta}. \tag{8}$$

More importantly, $p = (x_0, y_0) \in W^{\lambda_1}$ *if and only if the orbit* $\gamma_p = \{f^n(p)\}_{n=0}^\infty$ *is a fixed point of* T *and*

$$p = (x_0, y_0) = \left(x_0, \sum_{i=1}^\infty {A_2}^{1-i} h_2(p_i)\right). \tag{9}$$

Proof. For the parameters satisfying (7), we can choose an adapted norm to satisfy the relations below

$$\|{A_2}^{-1}\| < \alpha,\ \|A_1\| < \nu < \beta < 1/\alpha < \|A_2\|. \tag{10}$$

We now show $\bar{\gamma} \in S_\beta$. Specifically, because $\|h_1(p)\| = \|h_1(p) - h_1(0)\| \leq L\|p\|$ and $\nu < \beta$, we have for $\bar{x}_n$

$$\begin{aligned} \|\bar{x}_n\| &\leq \|{A_1}^n\|\|x_0\| + \sum_{i=1}^n \|{A_1}^{n-i} h_1(p_{i-1})\| \\ &\leq \nu^n\|x_0\| + \sum_{i=1}^n \nu^{n-i} L\beta^{i-1}\|\gamma\|_\beta \\ &= \nu^n\|x_0\| + L\|\gamma\|_\beta \tfrac{\beta^n - \nu^n}{\beta - \nu} \leq (\|x_0\| + \tfrac{L\|\gamma\|_\beta}{\beta-\nu})\beta^n. \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} \|\bar{y}_n\| &\leq \sum_{i=n+1}^\infty \|{A_2}^{n+1-i} h_2(p_i)\| \leq \sum_{i=n+1}^\infty \alpha^{i-n-1} L\beta^i\|\gamma\|_\beta \\ &= \alpha^{-n-1} L\|\gamma\|_\beta \tfrac{(\alpha\beta)^{n+1}}{1-\alpha\beta} = \tfrac{L\beta\|\gamma\|_\beta}{1-\alpha\beta}\beta^n. \end{aligned} \tag{12}$$

Hence, T is well-defined and the bound estimate (8) holds, implying $T : S_\beta \to S_\beta$.

Next, for any $p := p_0 = (x_0, y_0) \in W^{\lambda_1}$, by definition $\gamma_p = \{p_n = f^n(p_0)\} \in S_\beta$, so $\|p_n\| \leq \|\gamma\|_\beta \beta^n$ for $n \geq 0$. Because for $m \geq n$, $\|{A_2}^{n-m}\| \leq \alpha^{m-n}$, and $\alpha\beta < 1$, the first term of the y_n-equation of the VPF (2) goes to 0 as $m \to \infty$. Because of the estimate (12), the partial sum of the y_n-equation converges as well as $m \to \infty$. So every orbit from W^{λ_1} satisfies

$$\left\{ \begin{array}{l} x_n = {A_1}^n x_0 + \sum_{i=1}^n {A_1}^{n-i} h_1(p_{i-1}) \\ y_n = \sum_{i=n+1}^\infty {A_2}^{n+1-i} h_2(p_i), \end{array} \right. \tag{13}$$

showing γ_p is a fixed point of T.

Conversely, if a sequence $\gamma = \{p_n = (x_n, y_n)\} \in S_\beta$ is a fixed point of T, satisfying (13), then it is straightforward to verify

$$x_{n+1} = A_1 x_n + h_1(x_n, y_n) \text{ and } y_n = {A_2}^{-1} y_{n+1} + h_2(x_{n+1}, y_{n+1})$$

hold for all $n \geq 0$. By (1) the sequence is an orbit of f. Therefore, $\gamma = \gamma_p \in W^{\lambda_1},\ p = (x_0, y_0)$ by definition. And equation (9) holds from (13). $\square$

Lemma 2. *There is a Lipschitz continuous function $\phi_2 \in C^{0,1}(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ so that*

$$W^{\lambda_1} = \mathrm{graph}(\phi_2). \tag{14}$$

Proof. By Lemma 1, we know that $p \in W^{\lambda_1}$ if and only if p is the initial point of a sequence $\gamma \in S_\beta$ which is a fixed point of the map T defined by (6) and (9) holds. To show the existence of such a fixed point, we will consider T as a map parameterized by $x_0 \in \mathbb{E}^{\lambda_1}$ and show that $T(\cdot, x_0) : S_\beta \to S_\beta,\ x_0 \in \mathbb{E}^{\lambda_1}$, is a uniform contraction. Specifically, let γ, γ' and $\bar\gamma = T(\gamma, x_0), \bar\gamma' = T(\gamma', x_0)$. We have

$$\begin{aligned} \|\bar x_n - \bar x_n'\| &\leq \textstyle\sum_{i=1}^n \|{A_1}^{n-i}[h_1(p_{i-1}) - h_1(p_{i-1}')]\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L \|p_{i-1} - p_{i-1}'\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} L \beta^{i-1} \|\gamma - \gamma'\|_\beta \\ &\leq \tfrac{L}{\beta - \nu} \beta^n \|\gamma - \gamma'\|_\beta \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|\bar y_n - \bar y_n'\| &\leq \textstyle\sum_{i=n+1}^\infty \|{A_2}^{n+1-i}[h_2(p_i) - h_2(p_i')]\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \alpha^{i-n-1} L \|p_i - p_i'\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \alpha^{i-n-1} \beta^i \|\gamma - \gamma'\|_\beta \\ &\leq \tfrac{L\beta}{1-\alpha\beta} \beta^n \|\gamma - \gamma'\|_\beta\, . \end{aligned} \tag{16}$$

Hence,

$$\|T(\gamma, x_0) - T(\gamma', x_0)\|_\beta \leq (\tfrac{L}{\beta-\nu} + \tfrac{L\beta}{1-\alpha\beta}) \|\gamma - \gamma'\|_\beta\, ,$$

showing $T(\cdot, x_0)$ is a uniform contraction provided

$$\theta := \theta(\beta) = \tfrac{L}{\beta-\nu} + \tfrac{L\beta}{1-\alpha\beta} < 1 \tag{17}$$

which is true for small $\|f - Df(\bar q)\|_1$ by (4). Denote the unique fixed point of $T(\cdot, x_0)$ by

$$\gamma^*(x_0) = \{p_n(x_0)\}_{n=0}^\infty,\ \ p_n(x_0) = (x_n(x_0), y_n(x_0)),\ n \geq 0. \tag{18}$$

Define

$$\phi_2(x_0) := y_0(x_0) = \textstyle\sum_{i=1}^\infty {A_2}^{1-i} h_2(p_i(x_0)), \tag{19}$$

the y-coordinate of the initial point of the fixed point $\gamma^*(x_0)$. By Lemma 1(9), we have $p \in W^{\lambda_1}$ iff $p = (x_0, y_0) = (x_0, \phi_2(x_0))$, i.e., the identity (14).

Next, since $T : S_\beta \times \mathbb{E}^{\lambda_1} \to S_\beta$ is Lipschitz continuous in x_0 with

$$\|T(\gamma, x_0) - T(\gamma, {x_0}')\|_\beta \leq \|x_0 - {x_0}'\|$$

because $\|{A_1}^n\| < \beta^n$, we have by the Uniform Contraction Principle I that $\gamma^*(x_0)$ is Lipschitz continuous with

$$\|\gamma^*(x_0) - \gamma^*(x_0{}')\|_\beta \leq \tfrac{1}{1-\theta}\|x_0 - x_0{}'\| \tag{20}$$

which in turn implies ϕ_2 is Lipschitz continuous with

$$\|\phi_2(x_0) - \phi_2(x_0{}')\| \leq \|\gamma^*(x_0) - \gamma^*(x_0{}')\|_\beta \leq \tfrac{1}{1-\theta}\|x_0 - x_0{}'\| \,,$$

completing the proof of the lemma. □

Lemma 3. *$\phi_2 \in C^1(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ and $\mathbb{T}_{\bar{q}}W^{\lambda_1} \cong \mathbb{E}^{\lambda_1}$.*

Proof. The main argument is to show that the Uniform Contraction Principle II applies to T for $k = 1$. Two conditions are needed to verify: (1) $T \in C^1(S_\beta \times \mathbb{E}^{\lambda_1}, S_\beta)$; and (2) $\|D_\gamma T(\gamma, x_0)\|$ is uniformly bounded by a constant smaller than 1.

To verify the conditions, let $\gamma = \{p_n\}, v = \{v_n\} \in S_\beta$, and formally differentiate (6). Then $D_\gamma T(\gamma, x_0)v$ needs to be as below in components:

$$\begin{cases} [D_\gamma T(\gamma, x_0)v]_{n,\,1} = \sum_{i=1}^{n} {A_1}^{n-i} Dh_1(p_{i-1})v_{i-1} \\ [D_\gamma T(\gamma, x_0)v]_{n,\,2} = \sum_{i=n+1}^{\infty} {A_2}^{n+1-i} Dh_2(p_i)v_i. \end{cases} \tag{21}$$

By exactly the same estimates as for (15, 16) we have

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\,1}\| \leq \tfrac{L}{\beta-\nu}\beta^n\|v\|_\beta$$

and

$$\|[D_\gamma T(\gamma, x_0)v]_{n,\,2}\| \leq \tfrac{L\beta}{1-\alpha\beta}\beta^n\|v\|_\beta \,.$$

These estimates imply three things. One, because of the uniform convergence of the second equation, the derivative $D_\gamma T(\gamma, x_0)$ is well-defined. Two, the derivative is in fact in $L(S_\beta, S_\beta)$ as required. Three, the derivative's β-norm

$$\|D_\gamma T(\gamma, x_0)\|_\beta \leq \theta(\beta) < 1$$

is bounded by the same uniform contraction constant $\theta(\beta)$. About its derivative in x_0, we have

$$[D_{x_0} T(\gamma, x_0)]_{n,\,1} = {A_1}^n, \text{ and } [D_{x_0} T(\gamma, x_0)]_{n,\,2} = 0.$$

Obviously, $D_{x_0}T(\gamma, x_0) \in L(\mathbb{E}^{\lambda_1}, S_\beta)$ since $\|{A_1}^n\| < \beta^n$. This shows the Uniform Contraction Principle II indeed applies for T with the case of $k = 1$. Thus, we can conclude that for the fixed point, $\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_\beta)$, and $\phi_2 \in C^1(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ follows.

Furthermore, since the fixed point $\bar{q} \sim 0$ is obviously on the manifold, we have $\gamma_0 = \gamma^*(0) = \{0\}_{n\geq 0}$, the zero sequence. Hence, $\phi_2(0) = 0$ because $h_2(0) = 0$. In addition, for the derivative of ϕ_2, we have from (19)

$$D\phi_2(x_0) = \sum_{i=1}^{\infty} {A_2}^{1-i} Dh_2(p_i(x_0))Dp_i(x_0).$$

Because $Dh_2(0) = 0$, and $p_i(0) = 0$ for all $i \geq 0$, we have

$$D\phi_2(0) = 0,$$

showing that the tangent space of W^{λ_1} at $\bar{q} \sim 0$ is the lambda-left eigenspace $\mathbb{E}^{\lambda_1}$. This proves the theorem for $k = 1$. □

Lemma 4. *The definition of W^{λ_1} is independent of any two choices in β. More specifically, let $\gamma^*(x_0)$ be the fixed point of the map $T(\cdot, x_0)$ from Lemma 1, then for any $1 < \beta' < \beta$, a sufficiently small $\|f - Df(\bar{q})\|_1$ implies $\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_{\beta'})$ and $\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_\beta)$.*

Proof. Let β' and β be two different constants satisfying the definition of W^{λ_1}. Assume without loss of generality that $\lambda_1 < \beta' < \beta < \lambda_2$. On one hand, it is automatically true by definition that

$$W_{\beta'}^{\lambda_1} \subseteq W_\beta^{\lambda_1}$$

because $S_{\beta'} \subset S_\beta$ for $\beta' < \beta$.

On the other hand, we can re-adjust the adapted norm if necessary so that

$$\|{A_2}^{-1}\| < \alpha, \;\; \|A_1\| < \nu < \beta' < \beta < 1/\alpha < \|A_2\|.$$

Also, by making $\|f - Df(\bar{q})\|_1$ smaller if necessary, we can assume

$$\theta(\beta'),\; \theta(\beta) < 1.$$

Thus, the same estimates (11, 12) imply that the uniform contraction map $T(\cdot, x_0)$ defined in S_β maps the subset $S_{\beta'}$ into itself. Therefore, the fixed point function $\gamma^*(\cdot)$ for parameter β must reside in $S_{\beta'}$, and therefore the reverse inclusion $W_\beta^{\lambda_1} \subseteq W_{\beta'}^{\lambda_1}$ follows, implying

$$W_{\beta'}^{\lambda_1} = W_\beta^{\lambda_1}\;,$$

i.e., the independence of W^{λ_1} on β. The proof of Lemma 3 also shows the same fixed point function $\gamma^*(\cdot)$ is in both $C^1(\mathbb{E}^{\lambda_1}, S_{\beta'})$ and $C^1(\mathbb{E}^{\lambda_1}, S_\beta)$. □

Lemma 5. *f is a uniform Lipschitz on W^{λ_1} and for the adapted norm from Lemma 1 the Lipschitz constant is $\leq \beta$.*

Proof. Let $p_0 = (x_0, \phi_2(x_0)), p_0' = ({x_0}', \phi_2({x_0}'))$ be two points from W^{λ_1}, and consider their images under f, $p_1 = f(p_0), p_1' = f(p_0')$. Because their orbits, $\gamma^*(x_0), \gamma^*({x_0}')$, are fixed points of T, by (13) and (10) we have

$$\begin{aligned}\|x_1 - x_1'\| &\leq \|A_1\|\|x_0 - {x_0}'\| + \|h_1(p_0) - h_1(p_0')\| \\ &\leq \nu\|x_0 - {x_0}'\| + L\|p_0 - p_0'\| \\ &\leq (\nu + L)\|p_0 - p_0'\|\end{aligned}$$

and by (13), (10), and (20)

$$
\begin{aligned}
\|y_1 - y_1'\| &\leq \sum_{i=2}^{\infty} \|A_2{}^{2-i}[h_2(p_i) - h_2(p_i')]\| \\
&\leq \sum_{i=2}^{\infty} \alpha^{i-2} L \|p_i - p_i'\| \\
&\leq L \sum_{i=2}^{\infty} \alpha^{i-2} \beta^i \|\gamma^*(x_0) - \gamma^*(x_0{}')\|_\beta \\
&\leq \tfrac{L\beta^2}{1-\alpha\beta} \tfrac{1}{1-\theta} \|x_0 - x_0{}'\| \\
&\leq \tfrac{L\beta^2}{1-\alpha\beta} \tfrac{1}{1-\theta} \|p_0 - p_0'\|.
\end{aligned}
$$

Hence,

$$
\|f(p_0) - f(p_0')\| \leq (\nu + L + \tfrac{L\beta^2}{1-\alpha\beta} \tfrac{1}{1-\theta}) \|p_0 - p_0'\| < \beta \|p_0 - p_0'\|
$$

for small L, i.e., for small $\|f - Df(\bar{q})\|_1$. □

Lemma 6. *If* ${\lambda_1}^k < \lambda_2$ *and* $f \in C^k(\mathbb{R}^d),\ 1 \leq k < \infty$*, then* $\phi_2 \in C^k(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$*. If* ${\lambda_1}^{k+1} < \lambda_2$ *and* $f \in C^{k,1}(\mathbb{R}^d)$*, then* $\phi_2 \in C^{k,1}(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$*.*

Proof. The $k = 1$ case is proved in Lemma 3. For $k \geq 2$, we note that the Uniform Contraction Principle II cannot apply directly as the proof of Lemma 3 did for $k = 1$. This is because we cannot prove $T \in C^k(S_\beta \times \mathbb{E}^{\lambda_1}, S_\beta)$. An indirect approach is needed. We consider the C^k case first in details because the $C^{k,1}$ case follows easily.

Because of the assumption ${\lambda_1}^k < \lambda_2$, we can choose ς close to λ_1 and β close to λ_2 so that the following conditions hold

$$
\lambda_1 < \varsigma < \beta < \lambda_2, \text{ and } {\lambda_1}^k < \varsigma^k < \beta < \lambda_2. \tag{22}
$$

And assume

$$
\|A_1\| < \nu < \varsigma < \beta < 1/\alpha,\ \|{A_2}^{-1}\| < \alpha < 1, \tag{23}
$$

by re-adjusting the adapted norm if necessary. By Lemma 4, we have for small $\|f - Df(\bar{q})\|_1$ and $\beta' = \varsigma$ the following

$$
\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_\varsigma) \text{ and } T \in C^1(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\varsigma). \tag{24}
$$

We want to prove first instead the following claim

$$
T \in C^k(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta). \tag{25}
$$

We note first that

$$
[D_{x_0} T(\gamma, x_0)]_{n,\,1} = {A_1}^n, \text{ and } [D_{x_0} T(\gamma, x_0)]_{n,\,2} = 0.
$$

This implies any mixed derivative in γ and x_0 are the zero operators, hence well-defined and exists. So, we only need to show T is $C^k(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta)$ separately in γ and x_0. For the latter, the identity above shows

$$
\|[D_{x_0} T(\gamma, x_0)]_n\| \leq \|{A_1}^n\| < \nu^n < \beta^n
$$

and $\|D_{x_0}T(\gamma, x_0]\|_\beta \leq 1$ follows. Also, $D^j_{x_0}T(\gamma, x_0) = 0$, for $2 \leq j \leq k$. Hence, $T(\gamma, \cdot) \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$.

Now we show $T(\cdot, x_0) \in C^k(S_\varsigma, S_\beta)$, i.e., $D^j_\gamma T(\gamma, x_0)$ is a bounded j-linear form from $\otimes^j S_\varsigma$ to S_β for any $1 \leq j \leq k$. The case of $j = 1$ is true by (24) because $T(\cdot, x_0) \in C^1(S_\varsigma, S_\varsigma) \subset C^1(S_\varsigma, S_\beta)$ since $S_\varsigma \subset S_\beta$ for $\varsigma < \beta$.

For any $2 \leq j \leq k$, $[D^j_\gamma T(\gamma, x_0)]$ should be a bounded j-linear form from S_ς to S_β. To this end, let $v = v^1 \otimes v^2 \otimes \cdots \otimes v^j$ with each $v^\ell \in S_\varsigma$. Formally differentiate (6) to get

$$\begin{cases} [D^j_\gamma T(\gamma, x_0)v]_{n,\,1} = \sum_{i=1}^n {A_1}^{n-i} D^j h_1(p_{i-1}) v_{i-1} \\ [D^j_\gamma T(\gamma, x_0)v]_{n,\,2} = \sum_{i=n+1}^\infty {A_2}^{n+1-i} D^j h_2(p_i) v_i, \end{cases} \tag{26}$$

where

$$v_i = v_i^1 \otimes v_i^2 \otimes \cdots \otimes v_i^j, \quad v_i^\ell \in \mathbb{R}^d.$$

Similar to the estimate of (15), we have

$$\begin{aligned} \|[D^j_\gamma T(\gamma, x_0)v]_{n,\,1}\| &\leq \textstyle\sum_{i=1}^n \|{A_1}^{n-i}\| \|[D^j h_1(p_{i-1})]v_{i-1}\| \\ &\leq \textstyle\sum_{i=1}^n \nu^{n-i} \|h_1\|_j \Pi_{\ell=1}^j \|v_{i-1}^\ell\| \\ &\leq \|h_1\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \varsigma^{j(i-1)} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \|h_1\|_k \textstyle\sum_{i=1}^n \nu^{n-i} \beta^{i-1} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \tfrac{\|h_1\|_k}{\beta-\nu} \beta^n \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \end{aligned} \tag{27}$$

where $\|A_1\| < \nu < \varsigma < \beta$ and $\varsigma^k < \beta$ by (22, 23), which imply $\varsigma^j < \beta$ for $1 \leq j \leq k$. Similar to the estimate of (16) we have

$$\begin{aligned} \|[D^j_\gamma T(\gamma, x_0)v]_{n,\,2}\| &\leq \textstyle\sum_{i=n+1}^\infty \|{A_2}^{n+1-i}\| \|[D^j h_2(p_i)]v_i\| \\ &\leq \textstyle\sum_{i=n+1}^\infty \alpha^{i-n-1} \|h_2\|_j \varsigma^{ji} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \|h_2\|_k \alpha^{-n-1} \textstyle\sum_{i=n+1}^\infty (\alpha\varsigma^j)^i \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \|h_2\|_k \alpha^{-n-1} \tfrac{(\alpha\beta)^{n+1}}{1-\alpha\beta} \Pi_{\ell=1}^j \|v^\ell\|_\varsigma \\ &\leq \tfrac{\|h_2\|_k \beta}{1-\alpha\beta} \beta^n \Pi_{\ell=1}^j \|v^\ell\|_\varsigma. \end{aligned} \tag{28}$$

Combine these two estimates to obtain

$$\|[D^j_\gamma T(\gamma, x_0)]\|_\beta \leq \|(h_1, h_2)\|_k \max\{\tfrac{1}{\beta-\nu}, \tfrac{\beta}{1-\alpha\beta}\}.$$

The convergence of the infinite series also shows the derivatives are well-defined. This completes the proof that $T \in C^k(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta)$.

We are now ready to show $\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$. By the Uniform Contraction Principle II for $T \in C^1(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\varsigma)$, the fixed point $\gamma^*(\cdot)$ is in $C^1(\mathbb{E}^{\lambda_1}, S_\varsigma)$ and its derivative is given by

$$D\gamma^*(\cdot) = \textstyle\sum_{n=0}^\infty [D_\gamma T(\gamma^*(\cdot), \cdot)]^n D_{x_0} T(\gamma^*(\cdot), \cdot).$$

Since $\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_\varsigma)$, $T \in C^1(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\varsigma) \subset C^1(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta)$, and $T \in C^k(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta), k \geq 2$, here is the key to notice that the composition

$D_\gamma T(\gamma^*(\cdot),\cdot)$ is $C^1(\mathbb{E}^{\lambda_1}, S_\beta)$. This implies that the infinite series on the right is in $C^1(\mathbb{E}^{\lambda_1}, S_\beta)$, and therefore, $D\gamma^*(\cdot) \in C^1(\mathbb{E}^{\lambda_1}, S_\beta)$, and $\gamma^*(\cdot) \in C^2(\mathbb{E}^{\lambda_1}, S_\beta)$ follows. Apply this argument recursively to obtain $\gamma^*(\cdot) \in C^3(\mathbb{E}^{\lambda_1}, S_\beta)$, and so on until we reach $\gamma^*(\cdot) \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$. As a component of the initial point of γ^*, ϕ_2 is in $C^k(\mathbb{E}^{\lambda_1}, \mathbb{E}^{\lambda_2})$ as well.

For the case of $f \in C^{k,1}$, the argument above can be used to show first $T \in C^{k,1}(S_\varsigma \times \mathbb{E}^{\lambda_1}, S_\beta)$, using ${\lambda_1}^{k+1} < \varsigma^{k+1} < \beta < \lambda_2$, and then $\gamma^* \in C^{k,1}(\mathbb{E}^{\lambda_1}, S_\beta)$, which in turn implies ϕ_2 is $C^{k,1}$. This completes the proof. $\square$

The lemmas above complete the proof for Theorem 1. For future reference, we state the following result from the proofs above.

Proposition 1. *Let $[\lambda_1, \lambda_2]$ be a pseudo-hyperbolic split of $J = Df(\bar{q})$ for a diffeormorphism f in $\mathbb{R}^d$ at a fixed point $\bar{q}$. For any $\lambda_1 < \varsigma < \beta < \lambda_2$ and small $\|f - Df(\bar{q})\|_1$, the orbit $\gamma_p = \{f^n(p)\}_{n=0}^{\infty}$ of any point $p = (x_0, y_0) \in W^{\lambda_1}$ can be expressed as a function $\gamma_p = \gamma^*(x_0)$ for $x_0 \in \mathbb{E}^{\lambda_1}$ so that $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\varsigma)$ and $\gamma^* \in C^k(\mathbb{E}^{\lambda_1}, S_\beta)$ if ${\lambda_1}^k < \lambda_2$ and $f \in C^k(\mathbb{R}^d), 1 \le k < \infty$, or $\gamma^* \in C^{k,1}(\mathbb{E}^{\lambda_1}, S_\varsigma)$ and $\gamma^* \in C^{k,1}(\mathbb{E}^{\lambda_1}, S_\beta)$ if ${\lambda_1}^{k+1} < \lambda_2$ and $f \in C^{k,1}(\mathbb{R}^d)$.*