Applications of IFT: Inverse Function Theorems

Theorem 1 (Global Inverse Function Theorem)**.** *Let $A_{n\times n}$ be nonsingular and $h \in C^1(\mathbb{R}^n)$. Then there is a small number $\delta > 0$ so that $\sup_{x\in\mathbb{R}^n}(|h(x)| + |Dh(x)|) < \delta$ implies $f(x) = Ax + h(x)$ is invertible and the inverse f^{-1} is as smooth as f. Moreover, f^{-1} can be expressed as $f^{-1} = A^{-1} + g$ with $g = -A^{-1} \circ h \circ f^{-1}$, $\sup_{x\in\mathbb{R}^n}(|g(x)| + |Dg(x)|) \le \epsilon$ and $\lim_{\delta\to 0} \epsilon = 0$. Furthermore, if f is C^k for $k \ge 1$ or analytic then f^{-1} is also C^k or analytic, respectively.*

Proof. Let $X = C^1(\mathbb{R}^n)$ be the Banach space of functions from $\mathbb{R}^n$ to itself for which they and their derivatives are uniformly continuous and uniformly bounded with norm

$$||h||_1 = \sup_{x\in\mathbb{R}^n} (|h(x)| + |Dh(x)|).$$

We look for inverse of the form $\phi = A^{-1} + g$ with $g \in X$

$$\mathrm{id} = \phi \circ f = (A^{-1} + g) \circ (A + h) = \mathrm{id} + A^{-1} \circ h + g \circ (A + h)$$

equivalent to

$$F(g, h) := A^{-1} \circ h + g \circ (A + h) = 0.$$

Obviously, $F(g, h) \in X$, showing $F : X \times X \to X$. Also, F is differentiable in g, h with $D_gF(g, h)v = v \circ (A + h)$ and $D_hF(g, h)v = Dg \circ (A + h)v$ for any $v \in X$, showing $F \in C^1(X \times X, X)$. Moreover, $D_gF(0, 0)v = v \circ A = w$ for any $w \in X$ iff $v = w \circ A^{-1}$. This shows $D_gF(0, 0) \in L(X, X)$ is invertible with a bounded inverse since $v = [D_gF(0, 0)]^{-1}w = w \circ A^{-1}$ and $|[D_gF(0, 0)]^{-1}| = 1$. Since in addition $F(0, 0) = 0$, therefore, by IFT there are open neighborhood $V = N_{\delta_1}(0), U = N_{\delta_2}(0) \subset X$ for some small numbers $\delta_1, \delta_2 > 0$ and a $u \in C^1(V, U)$ so that $F(g, h) = 0$ for $(g, h) \in U \times V$ iff $g = u(h)$. So, the left-inverse $\phi(h) = A^{-1} + u(h)$ exists and is of C^1.

To show ϕ is also the right-inverse, consider similarly the right-inverse of the form $\psi = A^{-1} + g$ with

$$\mathrm{id} = f \circ \psi = (A + h) \circ (A^{-1} + g) = \mathrm{id} + A \circ g + h \circ (A^{-1} + g)$$

equivalent to

$$G(g, h) := A \circ g + h \circ (A^{-1} + g) = 0.$$

It is similar to show $G \in C^1(X \times X, X)$ and $G(0, 0) = 0$. It is slightly different to show $D_gG(0, 0)$ has a bounded inverse. Specifically, for any $v \in X$,

$$D_gG(0, 0)v = [A + Dh(A^{-1}\cdot)]v = A[\mathrm{id} + A^{-1}Dh(A^{-1}\cdot)]v,$$

which means

$$[D_gG(0, 0)v](x) = [A + Dh(A^{-1}x)]v(x) = A[\mathrm{id} + A^{-1}Dh(A^{-1}x)]v(x).$$

So $D_g G(0,0)$ is invertible if $T \in L(X,X)$ with $T(x) = A^{-1}Dh(A^{-1}x)$ is bounded by $\sup_{x\in\mathbb{R}^n} |T(x)| < 1$ which holds if $\sup_{x\in\mathbb{R}^n} |Dh(x)| < 1/|A^{-1}| := r$. So if we let $Y = \bar{N}_r(0) \subset X$, then for $G \in C^1(X \times Y, X)$, $D_g G(0,0) \in L(X,X)$ has a bounded inverse. Therefore, by IFT there are open neighborhood $V = N_{\delta_1'}(0) \subset Y, U = N_{\delta_2'}(0) \subset X$ for some small numbers $\delta_1', \delta_2' > 0$ and a $w \in C^1(V,U)$ so that $G(g,h) = 0$ for $(g,h) \in U \times V$ iff $g = w(h)$. That is, the right-inverse $\psi(h) = A^{-1} + w(h)$ exists.

Next, to show ϕ and ψ are the same function, let $\delta = \min\{\delta_1, \delta_1'\}$, and $\gamma = \max\{\delta_2, \delta_2'\}$, then both u and w map $V = N_\delta(0) \subset X$ to $U = N_\gamma(0) \subset X$. Because of the continuity, $\lim_{\delta\to 0} \epsilon = 0$ where $\epsilon = \max\{\|u\|_1, \|w\|_1\}$. As a result, both $\phi(h) = A^{-1} + u(h)$ and $\psi(h) = A^{-1} + w(h)$ are defined for $h \in V$ so that $\phi(h) \circ f = \mathrm{id}$ and $f \circ \psi(h) = \mathrm{id}$ imply

$$\phi(h) = \phi(h) \circ \mathrm{id} = \phi(h) \circ (f \circ \psi(h)) = \psi(h)$$

by the associative law of composition. By definition, we have $\phi(h) = f^{-1}$.

Finally, if h is C^k for $k \geq 1$ or analytic, then both F and G have the same smoothness, and by IFT both ϕ and ψ have the same smoothness as well. As a consequence, f^{-1} is as smooth as h is. □

Lemma 1 (Cut-off Function)**.** *For each $r > 0$ there exists a C^∞ function $\rho_r : \mathbb{R}^n \to [0,1]$ so that $\rho_r|_{N_r} \equiv 1$ and $\mathrm{supp}\{\rho\} \subset N_{2r}$, where N_r is the Euclidean ball of radius r in $\mathbb{R}^n$ centered at 0.*

Proof. Let $|x| = \sqrt{\sum x_i^2}$ be the Euclidean norm for $\mathbb{R}^n$. Define

$$\phi(x) = \begin{cases} \exp(-1/(1-4|x|^2), & |x| < 1/2 \\ 0, & 1/2 \leq |x| \end{cases}$$

Then ϕ is a C^∞ function with support $\mathrm{supp}\{\phi\} \subset N_{1/2}$. Let

$$a = \textstyle\int_{\mathbb{R}^n} \phi(x)dx,$$

which is a positive number. Let

$$\chi(x) = \begin{cases} 1, & |x| < 3/2 \\ 0, & 3/2 \leq |x| \end{cases}$$

be the characteristic function of the radius-3/2 ball $N_{3/2}$ of 0. Define

$$\rho_1(x) = \tfrac{1}{a}\phi * \chi(x) = \tfrac{1}{a}\textstyle\int_{\mathbb{R}^n} \phi(x-y)\chi(y)dy$$

where $\phi * \chi$ is the convolution of ϕ and χ. The integral exists because both functions have a finite support. Also ρ_1 is as smooth as ϕ is. In addition, for any $x \in N_1$, and $x - y$ in the support of ϕ with $|x-y| < 1/2$, we have that y is in the support of χ because $|y| \leq |x| + |x-y| \leq 3/2$. So

$$\rho_1(x) = \tfrac{1}{a}\textstyle\int_{\mathbb{R}^n} \phi(x-y)\chi(y)dy = \tfrac{1}{a}\int_{\mathbb{R}^n} \phi(x-y)dy = 1.$$

On the other hand, for $|x| > 2$ and $x - y$ in the support of ϕ with $|x - y| < 1/2$, y is outside the support of χ because $|y| \geq |x| - |x - y| > 3/2$. Therefore

$$\rho_1(x) = \frac{1}{a} \int_{\mathbb{R}^n} \phi(x - y)\chi(y)dy = 0.$$

Clearly we also have $0 \leq \rho_1(x) \leq 1$. Last, for each $r > 0$, the required function is

$$\rho_r(x) = \rho_1(x/r).$$

This completes the proof. □

Theorem 2 (Local Inverse Function Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^k function for $k \geq 1$. Assume at a point x_0, $Df(x_0)$ is invertible. Then there is a small open neighborhood U of x_0, a small open neighborhood V of $y_0 = f(x_0)$ so that $f : U \to V$ is 1-1, onto, and the inverse f^{-1} is also C^k.*

Proof. First we claim that $f : U \to V$ is invertible iff $g : U' = U \oplus \{-x_0\} \to V' = V \oplus \{-y_0\}$ is invertible where $\bar{y} = g(\bar{x}) = f(\bar{x}+x_0)-y_0$, $\bar{x} = x-x_0 \in U'$. This can be checked directly as follows. Specifically, if f is invertible with inverse f^{-1}, then $g^{-1}(\bar{y}) = f^{-1}(\bar{y} + y_0) - x_0$ because

$$g \circ g^{-1}(\bar{y}) = f(g^{-1}(\bar{y}) + x_0) - y_0 = (\bar{y} + y_0) - y_0 = \bar{y},$$

and similarly $g^{-1} \circ g(\bar{x}) = \bar{x}$. If g is invertible with inverse g^{-1}, then $f^{-1}(y) = g^{-1}(y - y_0) + x_0$ because

$$f \circ f^{-1}(y) = [f(g^{-1}(y - y_0) + x_0) - y_0] + y_0 = g \circ g^{-1}(\bar{y}) + y_0 = y,$$

and similarly $f^{-1} \circ f(x) = x$.

So, without loss of generality, we can assume $x_0 = y_0 = 0$ for $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$. Now, let $A = Df(0)$, $k(x) = f(x) - Ax$. Then $k(0) = 0$, $Dk(x) = Df(x) - A$ and $Dk(0) = 0$. So by the continuous differentiability of f for any $\delta_1 > 0$ there is a small r-ball N_r of $x = 0$ so that

$$\sup_{x \in N_r} (|k(x)| + |Dk(x)|) \leq \delta_1.$$

Let ρ_r be a cut-off function from the previous lemma. Define

$$h(x) = \rho_{r/2}(x)k(x).$$

Then the support of h is inside N_r, and

$$|Dh(x)| = |D\rho_{r/2}(x)k(x) + \rho_{r/2}(x)Dk(x)| \leq K\delta_1$$

for a constant K and all $x \in \mathbb{R}^n$. Hence,

$$\sup_{x \in \mathbb{R}^n} (|h(x)| + |Dh(x)|) \leq (K + 1)\delta_1 := \delta$$

Therefore, by the Global Inverse Function Theorem, for sufficiently small $r > 0$, $F(x) = Ax + h(x)$ is C^k invertible in $\mathbb{R}^n$. For $x \in N_{r/2}$, since $\rho_{r/2}(x) \equiv 1$, we have $F(x) = Ax + h(x) = Ax + k(x) = f(x)$. Hence f is locally invertible from $U = N_{r/2}$ to $V = F(U)$, and the inverse, $f^{-1} = F^{-1}|_V$, is also C^k. □

Reference: S.-N. Chow and J.K. Hale, Methods of Bifurcation Theory, Springer-Verlag, 1982.