ARTICLE IN PRESS YJDEQ:6328

J. Differential Equations ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Neural spike renormalization. Part II — Multiversal chaos

Bo Deng

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, United States

ARTICLE INFO

Article history:
Received 22 July 2010
Available online xxxx

Keywords:
Spike renormalization operator
Periodic orbit
Sensitive dependence on initial conditions
Dense orbit
Conjugate embedding
Chaotic dynamical systems

ABSTRACT

Reported here for the first time is a chaotic infinite-dimensional system which contains infinitely many copies of every deterministic and stochastic dynamical system of all finite dimensions. The system is the renormalizing operator of spike maps that was used in a previous paper to show that the first natural number 1 is a universal constant in the generation of metastable and plastic spike-bursts of a class of circuit models of neurons.

1. Introduction

A new class circuit models for excitable membranes was constructed in [2] by separating the transmembrane's ion current into active currents of ion pumps and passive currents of both the electromagnetic and the diffusive kinds. It was further demonstrated in [3] that the models exhibit not only many important and known phenomena found from the Hodgkin–Huxley type models [9,11,10,1,8] but also a new and unique phenomenon in metastable and plastic spike-bursts driven by the intracellular biochemical energy conversion via the ion pumps. It was demonstrated in the previous paper [5] that the metastable plasticity of the spike-bursts of different types of neural circuits can be described by the so-called isospiking bifurcation of spike-bursts and the bifurcation admits a universal constant in the same sense as Feigenbaum's renormalization [6,7] for the logistic map family except that the universal constant is the first natural number 1 and the renormalization operator is the neural spike renormalization group associated with the isospiking bifurcation.

More specifically, the prototypical family of maps for spike-bursts is $\psi_\mu : [0,1] \mapsto [0,1]$ with $\psi_\mu(x) = x + \mu$ if $0 \leqslant x < 1 - \mu$ and $\psi_\mu(x) = 0$ if $1 - \mu \leqslant x \leqslant 1$ for which $0 < \mu < 1$ is a parameter proportional to the total absolute current through neuron's ion pumps that in turn is related to its intracellular biochemical energy conversion [4]. It was demonstrated in the previous paper that

E-mail address: bdeng1@math.unl.edu.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.10.004

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

the generation of a burst of n spikes is through the isospiking bifurcation point, μ_n, which is proportional to $1/n$, about the energy expenditure amount per spike. It was further demonstrated that this scaling law, $\mu_n \sim 1/n$ can be explained by a renormalization argument. That is, in the space of renormalizable maps the renormalizing operator, $\mathcal{R}$, has the identity map $\psi_0(x) = x$, $x \in [0, 1]$, as a non-hyperbolic fixed point at which the operator has a 1-dimensional center-unstable manifold which is precisely the family ψ_μ, and the weakly-expanding eigenvalue is the first natural number 1. Parallel to Feigenbaum's universality for the period-doubling cascade to chaos [6,7], the neural universality implied by the limit $(\mu_{n+1} - \mu_n)/(\mu_n - \mu_{n-1}) \to 1$ is explained by the property that all 1-parameter families of neural spike maps converge to the fixed point's center-unstable manifold by the way of the so-called λ-lemma for the infinitely dimensional dynamical system $\mathcal{R}$.

In this paper we will continue the analysis of the spike renormalization but focus instead on its dynamics on the center-stable manifold of the fixed point ψ_0, and to prove in a series of propositions the following statements:

Theorem of multiverse chaos. *The spike renormalizing operator $\mathcal{R}$ is chaotic in a subset X_0 of the center-stable manifold of the fixed point ψ_0 in the sense that it has a dense set of periodic orbits*; *the property of sensitive dependence on initial conditions*; *and a dense orbit. Moreover, every finite dimensional dynamical system, deterministic or probabilistic, is conjugate to infinitely many subsystems of the operator inside the chaotic set X_0.*

At a cursory glance it seems that all infinitely dimensional systems arising from partial differential equations are excluded from the conjugate embedding to the renormalization $\mathcal{R}$, but in fact the opposite is the case. This is because any practical implementation and simulation of such a PDE always results in a discrete, finite dimensional system which in turn is part of $\mathcal{R}$ by the theorem. This is true regardless the precision with which such infinitely systems are approximated through discretization. Even for time-independent PDEs their finitely dimensional discretizations will be represented as stationary states of the renormalization $\mathcal{R}$. At a conceptual level, there are only finitely many atoms or elementary particles in the visible universe that we live in and that in turn is probably not infinitely divisible neither spatially nor temporally. Thus, any discrete temporal evolution of all particles in an essentially finitely grided universe is conceptually a finite system in dimension, which according to the theorem is just one of infinitely many subsystems of the neural renormalizing group. That is, if there were infinitely many parallel universes like ours as many physicists now believe, they would be part of the chaotic renormalizing group as well. It is in this sense that $\mathcal{R}$'s chaos is multiversal.

2. Neural spike renormalization

The rest of the paper is to prove the theorem in a sequence of propositions. Like the previous paper once the operator is defined all proofs are set up as straightforward verifications of statements requiring at most the preparation of the first year graduate course on analysis knowing concepts up to separate spaces. We now begin by recalling some preparatory results from [5] in this section. By definition, a spike map $g : [0, 1] \to [0, 1]$ of the unit interval satisfies the following conditions:

(a) There is a constant $c_0^g \in (0, 1]$ such that g is continuous everywhere except at $x = c_0^g$.
(b) g is strictly increasing in interval $[0, c_0^g]$.
(c) $g(x) \geqslant x$ for $x \in [0, c_0^g]$.
(d) The right limit $\lim_{x \to (c_0^g)^+} g(x)$ exists and $g(x) \leqslant g(0)$ for $c_0^g \leqslant x \leqslant 1$.

The set of all spike maps is denoted by Y and it is endowed with the L^1 norm, $\|g\| = \int_0^1 |g(x)|\,dx$. That is, Y is a subset of the $L^1[0, 1]$ Banach space and the norm $\|g - h\| = \int_0^1 |g(x) - h(x)|\,dx$ of the difference represents the average distance $|g(x) - h(x)|$ over the interval $[0, 1]$ between the maps. Sometime it is convenient to find the norm by thinking it as the area bounded between the graphs of the two functions.

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

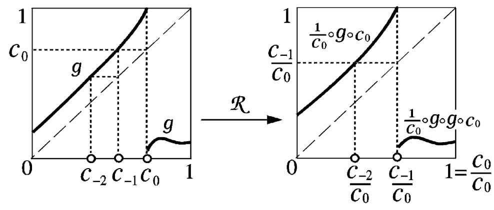

Fig. 1. A geometric illustration for $\mathcal{R}$.

Let $D = \{g \in Y\colon\ \exists c_{-1} \in (0, c_0)$ such that $g(c_{-1}) = c_0\}$. Then the *renormalizing operator* (or *renormalizing group*) $\mathcal{R} : D \to Y$ is defined as follows

$$g \in D \to \mathcal{R}[g](x) = \begin{cases} \frac{1}{c_0} g(c_0 x), & 0 \leqslant x < \frac{c_{-1}}{c_0}, \\ \frac{1}{c_0} g \circ g(c_0 x), & \frac{c_{-1}}{c_0} \leqslant x \leqslant 1. \end{cases}$$

That is, the graph of g over the subinterval $[0, c_{-1}]$ is scaled up to $[0, c_{-1}/c_0]$ and that over $[c_{-1}, c_0]$ is first composed with itself over $[c_0, 1]$ and then scale the composition in $[c_{-1}, c_0]$ to $[c_{-1}/c_0, 1]$. Both are scaled by the same factor $1/c_0$. Fig. 1 illustrates the operation graphically. A spike map from D is referred to as *renormalizable* and D is the *renormalizable* subset of Y.

It follows by definition of $\mathcal{R}$ and by induction that the following formula holds for any iterate of $\mathcal{R}$ whenever its previous iterate is renormalizable:

$$\mathcal{R}^k[g](x) = \begin{cases} \frac{1}{c_{-k+1}} g(c_{-k+1} x), & 0 \leqslant x < \frac{c_{-k}}{c_{-k+1}}, \\ \frac{1}{c_{-k+1}} g^{k+1}(c_{-k+1} x), & \frac{c_{-k}}{c_{-k+1}} \leqslant x \leqslant 1. \end{cases} \tag{1}$$

Here $c_{-k} = g^{-k}(c_0) \in [0, c_0)$ is the back iterates of c_0 by the renormalizable element g, which must exist for some $n \leqslant \infty$ and all $k = 1, 2, \ldots, n$ because of the renormalizable conditions (b), (c). More specifically, if c_0 has n backward iterates $c_{-k} = g^{-k} \in [0, c_0)$ for $k = 1, \ldots, n$, then the new point c_{-1}/c_0 which partitions the graph of $\mathcal{R}[g]$ into the parts above and below the diagonal has $n - 1$ backward iterates $c_{-j-1}/c_0 = \mathcal{R}[g]^{-j}(c_{-1}/c_0)$ in $[0, c_{-1}/c_0)$ for $j = 1, \ldots, n - 1$.

We further partition the spike map space Y as follows. Let

$$X = \bigl\{g \in D\colon\ \exists x_* \in [0, c_0],\ \text{such that } x_* = g\bigl(x_*^-\bigr)\bigr\}$$

where $g(x_*^-) = \lim_{x \to x_*^-} g(x)$. For $g \in X$, let $x_g = \sup\{x_* \in [0, c_0]\colon\ x_* = g(x_*^-)\}$, namely the right most fixed point, $x_g = g(x_g^-)$, of the map g in X. Then we define

$$X_0 = \{g \in X\colon\ x_g = 0\} \quad \text{and} \quad X_1 = \{g \in X\colon\ x_g > 0\}.$$

Naturally we have $X = X_0 \cup X_1$ and $g \in Y - X$ if and only if g does not have a fixed point in $[0, 1]$. It is trivial to note that the membership $g \in X_0$ forces $0 \leqslant g(x) \leqslant g(0) = 0$ for $x \in [c_0, 1]$.

We need to introduce for the remainder of the paper a so-called *concatenation* operation between two types of functions related to the set Y. Here is how. Let g be any function over an interval $[c_{-k-1}, c_0] \subset (0, 1)$ with these properties:

(i) $g(x) > x$ for $x \in [c_{-k-1}, c_0]$.
(ii) g is increasing.
(iii) $\{c_{-i}\}$ is the backward iterates of c_0: $g^{-i}(c_0) = c_{-i}$ or $g(c_{-i}) = c_{-i+1}$, $i = 1, 2, \ldots, k + 1$.

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

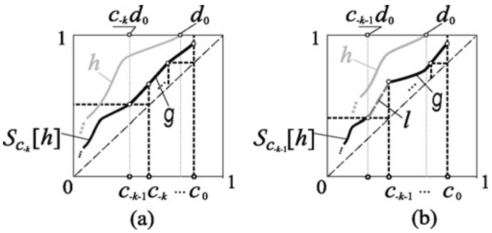

Fig. 2. A schematic illustration for the definition of the concatenation operation. (a) The case of $c_{-k}d_0 = c_{-k-1}$ for which $\mathcal{S}_{c_{-k}}[h]$ connects perfectly with g. (b) The other case of $c_{-k}d_0 \neq c_{-k-1}$ for which a line segment l is used to bridge the missing connection between $\mathcal{S}_{c_{-k-1}}[h]$ and g.

Let h be a function over an interval $[d_1, d_0] \subset [0, 1)$ such that $h(x) \geqslant x$, $x \in [d_1, d_0]$, and $h(d_0) = 1$. To define the concatenation, $g \vee h$, from g to h, we first scale down h by the factor of c_{-k}, and denote

$$\mathcal{S}_{c_{-k}}[h](x) := c_{-k} h\left(\frac{1}{c_{-k}} x\right), \quad c_{-k}d_1 \leqslant x \leqslant c_{-k}d_0.$$

For a pair of such functions g, h, the operation $\vee$ is defined depending on the following two situations: (a) If $c_{-k}d_0 = c_{-k-1}$, then we define

$$g \vee h(x) = \begin{cases} \mathcal{S}_{c_{-k}}[h](x), & c_{-k}d_1 \leqslant x \leqslant c_{-k}d_0 = c_{-k-1}, \\ g(x), & c_{-k-1} \leqslant x \leqslant c_0, \end{cases}$$

see Fig. 2(a). (b) If $c_{-k}d_0 \neq c_{-k-1}$, then we scale h down further

$$\mathcal{S}_{c_{-k-1}}[h](x) = c_{-k-1} h\left(\frac{1}{c_{-k-1}} x\right), \quad c_{-k-1}d_1 \leqslant x \leqslant c_{-k-1}d_0 < c_{-k-1}.$$

Since $c_{-k-1}d_0 < c_{-k-1}$ the domains of $\mathcal{S}_{c_{-k-1}}[h]$ and g do not overlap. Also since $\mathcal{S}_{c_{-k-1}}[h](c_{-k-1}d_0) = c_{-k-1} < c_{-k} = g(c_{-k-1})$, $\mathcal{S}_{c_{-k-1}}[h]$ lies below $y = c_{-k}$ and g lies above $y = c_{-k}$. We define $g \vee h$ by joining the points $(c_{-k-1}d_0, c_{-k-1})$ and (c_{-k-1}, c_{-k}) in the box $[0, 1] \times [0, 1]$ by a line denoted by l. That is, we define

$$g \vee h(x) = \begin{cases} \mathcal{S}_{c_{-k-1}}[h](x), & c_{-k-1}d_1 \leqslant x \leqslant c_{-k-1}d_0 < c_{-k-1}, \\ l(x), & c_{-k-1}d_0 \leqslant x \leqslant c_{-k-1}, \\ g(x), & c_{-k-1} \leqslant x \leqslant c_0, \end{cases}$$

with

$$l(x) = \frac{c_{-k} - c_{-k-1}}{c_{-k-1} - c_{-k-1}d_0}(x - c_{-k-1}d_0) + c_{-k-1},$$

see Fig. 2(b). The following properties will be used for the remainder of the paper.

Lemma 1.

(1) *$(g \vee h)(x) \geqslant x$ and $g \vee h$ is continuous in the domain of definition if both g and h are.*
(2) *If h satisfies properties* (i)–(iii) *as g does, then so does $g \vee h$.*
(3) *$g \vee h$ can be extended to be an element in X_0, and $\mathcal{R}^i[g \vee h] = h$ either $i = k$ or $i = k + 1$.*
(4) *The operation is associative, i.e., $g \vee (h \vee f) = (g \vee h) \vee f$, and we will denote $g \vee h \vee f = g \vee (h \vee f)$.*

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

(5) *If all defined, let* $\bigvee_{i=1}^{\infty} g_i = \lim_{n\to\infty} \bigvee_{i=1}^{n} g_i = g_1 \vee g_2 \vee g_3 \cdots$. *Then it holds that*

$$\lim_{x\to 0^+}\left(\bigvee_{i=1}^{\infty} g_i\right)(x) = 0.$$

Proof. For property (1)(a), it is straightforward to check in this case that $g \vee h(x) \geqslant x$ and $g \vee h$ is continuous as $\mathcal{S}_{c_{-k}}[h](c_{-k-1}) = c_{-k}h(\frac{1}{c_{-k}}c_{-k}d_0) = c_{-k}\cdot 1 = g(c_{-k-1})$. It is also the case for (b) as well by construction. Property (2) also follows from the construction. For property (3), it is based on the fact that the scaling down operation $\mathcal{S}_c[h]$ and the scaling up operation $\mathcal{R}$ are inverse operations of each other. More specifically, we have by definition, $g \vee h(x) = g(x)$, $x \in [c_{-k-1}, c_0]$, and

$$\mathcal{R}^i[g \vee h](x) = h(x) \quad \text{over } x \in [d_1, d_0]$$

for $i = k$ if $c_{-k}d_0 = c_{-k-1}$ and $i = k+1$ if $c_{-k}d_0 \neq c_{-k-1}$. Property (4) is also obvious. For property (5), we have that for any $n \geqslant 1$, the function $\bigvee_{i=1}^{n} g$ is defined over an interval $[a_n, c_0]$ whose left end point a_n is bounded from above by $c_{-k-1}c_{-k}^{n-1}$ by the definition of $\vee$. Thus, $a_n \to 0$ as $n \to \infty$ and $\lim_{x\to 0^+} \bigvee_{i=1}^{\infty} g_i(x) = 0$ follows by the monotonicity of the infinite concatenation. □

3. Multiverse chaos

We begin by pointing out the structure of X_1 is extremely simple. It is almost completely described by the following three propositions.

Proposition 1. *For any point* $g \in X_1$, *either* $x_g = 1$ *for which* g *is a fixed point of* $\mathcal{R}$ *or* $0 < x_g < 1$ *for which the orbit of* g *converges to a fixed point of* $\mathcal{R}$ *in* X_1.

Proof. Because of the monotonicity of g in $[0, c_0]$, the backward iterative sequence c_{-k} converges to x_g from above, $c_{-k} \searrow x_g$. By (1), $x_{g^k} = c_{-k}/c_{-k+1} \to x_g/x_g = 1 = x_{g^\infty}$ for which the limit $\lim g^k = g^\infty$ is a fixed point in X_1. □

Proposition 2. *For any* $0 < \rho < 1$, *there is an element* $s_\rho \in X_1$ *such that the orbit* $\{\mathcal{R}^n[s_\rho]\}$ *converges to the fixed point* $id = \psi_0$ *at the given rate* ρ.

Proof. This is done by construction. For each $0 < \rho < 1$, let

$$s_\rho(x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{\rho}(x - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} \leqslant x \leqslant \frac{1+\rho}{2}, \\ 0, & \frac{1+\rho}{2} < x \leqslant 1. \end{cases}$$

Thus, we have $c_0 = (1+\rho)/2$, and $x = 1/2$ is the largest fixed point of s_ρ. Since s_ρ is increasing with slope $1/\rho$ in $[1/2, (1+\rho)/2]$, $c_{-k} = s_\rho^{-k}(c_0)$ exists for all k and $c_{-k} \searrow 1/2$ as $k \to \infty$. Also because s_ρ is linear in $[1/2, (1+\rho)/2]$, we have

$$\frac{1}{\rho}(c_{-k} - c_{-(k+1)}) = (c_{-k+1} - c_{-k})$$

for all $k \geqslant 0$, with $c_0 = (1+\rho)/2$ and extending the notation to $c_1 = 1$. Solving this equation gives

$$c_{-k} = \frac{1}{2}\left[\rho^{k+1} + 1\right] \quad \text{for } k \geqslant 0 \quad \text{and} \quad c_{-k} \to \frac{1}{2} \quad \text{as } k \to \infty.$$

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

By (1),

$$\mathcal{R}^k[s_\rho](x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{1}{2c_{-k+1}}, \\ \frac{1}{\rho}(x - \frac{1}{2c_{-k+1}}) + \frac{1}{2c_{-k+1}}, & \frac{1}{2c_{-k+1}} \leqslant x \leqslant \frac{c_{-k}}{c_{-k+1}}, \\ 0, & \frac{c_{-k}}{c_{-k+1}} < x \leqslant 1. \end{cases}$$

It is easy to see $\mathcal{R}^k[s_\rho] \to id$ as $k \to \infty$. To demonstrate the convergence rate, we consider $\|\mathcal{R}^k[s_\rho] - id\|$ which consists of calculating the area of the triangle between s_ρ and the diagonal over the interval $[1/(2c_{-k+1}), c_{-k}/c_{-k+1}]$ and the trapezoid bounded between the diagonal and the x-axis over the interval $[c_{-k}/c_{-k+1}, 1]$. Therefore,

$$\begin{aligned} \|\mathcal{R}^k[s_\rho] - id\| &= \frac{1}{2}\left[\frac{c_{-k}}{c_{-k+1}} - \frac{1}{2c_{-k+1}}\right]\left[1 - \frac{c_{-k}}{c_{-k+1}}\right] + \frac{1}{2}\left[1 - \frac{c_{-k}}{c_{-k+1}}\right]\left[1 + \frac{c_{-k}}{c_{-k+1}}\right] \\ &= \frac{1}{2}\frac{\rho^k(1-\rho)}{1+\rho^k}\left[2\frac{c_{-k}}{c_{-k+1}} - \frac{1}{2c_{-k+1}}\right] \sim O\left(\rho^k\right) \quad \text{as } k \to \infty. \qquad \square \end{aligned}$$

Remark. This result can be extended to any fixed point in $g \in X_1$ with $x_g = 1$. We only need to modify in the construction above by using $\frac{1}{2}g(2x)$ for x in the leftmost interval $[0, 1/2]$. The only difference is in the estimation of the convergence rate which is not as straightforward as for the identity fixed point id in the proof above.

Proposition 3. *For every $\lambda > 1$, there exist fixed points $r_\lambda \in X_0$, $r_{1/\lambda} \in X_1$ and backward invariant families U_μ, V_μ with $U_0 = r_\lambda, V_0 = r_{1/\lambda}$ so that $\mathcal{R}$ expands at the rate of λ on both U_μ and V_μ.*

Proof. This is proved similarly as Proposition 2. Consider first the r_λ case. For each $\lambda > 1$ define

$$r_\lambda(x) = \begin{cases} \lambda x, & 0 \leqslant x \leqslant \frac{1}{\lambda}, \\ 0, & \frac{1}{\lambda} < x \leqslant 1. \end{cases}$$

It is straightforward to check that $c_0 = 1/\lambda$, $c_{-k} = 1/\lambda^{k+1}$, and that r_λ is a fixed point of $\mathcal{R}$, $\mathcal{R}[r_\lambda] = r_\lambda$, because of the linearity of r_λ. Define a family of renormalizable maps to be

$$U_\mu(x) = \begin{cases} \mu + \lambda x, & 0 \leqslant x \leqslant \frac{1-\mu}{\lambda}, \\ 0, & \frac{1-\mu}{\lambda} < x \leqslant 1. \end{cases}$$

It is the same ray as r_λ but translated upward by μ amount and clipped at $c_0 = (1-\mu)/\lambda$. Again, since U_μ is linear in $[0, (1-\mu)/\lambda]$, it is easy to check that U_μ is backward invariant with $\mathcal{R}[U_\mu] = U_{\frac{\lambda\mu}{1-\mu}}$ and $U_0 = r_\lambda$. To show the expanding rate, we first derive a formula for the distance $\|U_\mu - U_0\|$ which consists of the area of the parallelogram between U_μ and U_0 over the interval $[0, (1-\mu)/\lambda]$ and the area of the trapezoid between U_μ and U_0 over the interval $[(1-\mu)/\lambda, 1/\lambda]$. Namely,

$$\|U_\mu - U_0\| = \mu\frac{1-\mu}{\lambda} + \frac{1}{2}\left[\frac{1}{\lambda} - \frac{1-\mu}{\lambda}\right]\left[1 + \lambda\frac{1-\mu}{\lambda}\right] = \frac{\mu}{\lambda}\left[2 - \frac{3\mu}{2}\right].$$

Therefore

$$\frac{\|\mathcal{R}[U_\mu]-\mathcal{R}[U_0]\|}{\|U_\mu-U_0\|}=\frac{\lambda}{1-\mu}\frac{2-\frac{3}{2}\frac{\lambda\mu}{1-\mu}}{2-\frac{3}{2}\mu}\to\lambda,\quad \text{as } \mu\searrow 0.$$

This proves that $\mathcal{R}$ expands along U_μ at $U_0=r_\lambda$ at the rate of λ. A similar proof can be constructed for the $r_{1/\lambda}$ case. Specifically, we define

$$r_{1/\lambda}(x)=\frac{1}{\lambda}(x-1)+1\quad\text{and}\quad V_\mu(x)=\begin{cases} r_{1/\lambda}(x)+\mu, & 0\leqslant x\leqslant 1-\lambda\mu,\\ 0, & 1-\lambda\mu<x\leqslant 1.\end{cases}$$

It is similar to verify these identities: $V_0=r_{1/\lambda}$, $\mathcal{R}[V_\mu]=V_{\frac{\lambda\mu}{1-\lambda\mu}}$, $\|V_\mu-V_0\|=\mu(1+\lambda)+\frac{1}{2}\lambda\mu^2$, and as a consequence $\|\mathcal{R}[V_\mu]-\mathcal{R}[V_0]\|/\|V_\mu-V_0\|\to\lambda$ as $\mu\searrow 0$, which completes the proof. □

We note that the proof above in fact applies to the case with $\lambda=1$, reducing it to the special family of renormalizable maps

$$W_{id}^u:=\{\psi_\mu\colon 0\leqslant\mu\leqslant 1/2\}\quad\text{with } \psi_\mu(x)=\begin{cases}\mu+x, & 0\leqslant x<1-\mu,\\ 0, & 1-\mu\leqslant x\leqslant 1.\end{cases}$$

As it was shown in [5], the family defines a weakly-expanding center-unstable set of $\mathcal{R}$ at the fixed point $id=\psi_0=r_1$, with the expanding rate $1/(1-\mu)$ because $\mathcal{R}[\psi_\mu]=\psi_{\mu/(1-\mu)}$ which in turn reduces to the corresponding non-hyperbolic eigenvalue $\lambda=1$ at $\mu=0$.

Proposition 4. *The set of periodic points of $\mathcal{R}$ in X_0 is dense in X_0.*

Proof. We need to show that for any $g\in X_0$, there is a sequence of periodic points p_k such that $p_k\to g$ as $k\to\infty$. To construct p_k, we begin with the fact that since $g(x)>x$ for $0<x\leqslant c_0$, $g(0)=0$, and g is increasing in $[0,c_0]$, thus $c_{-k}=g^{-k}(c_0)$ exists for all $k\geqslant 1$ and $c_{-k}\to 0$ as $k\to\infty$. If $g(c_0)=1$, then we let $\tilde g_k=g|_{[c_{-k-1},c_0]}$. If $g(c_0)<1$, then for large k we let

$$\tilde g_k(x)=\begin{cases} g(x), & c_{-k-1}\leqslant x\leqslant c_0-\frac{1}{k},\\ k[1-g(c_0-\frac{1}{k})](x-c_0)+1, & c_0-\frac{1}{k}\leqslant x\leqslant c_0.\end{cases}$$

That is, $\tilde g_k$ in this case is constructed to be g over $[0,c_0-1/k]$ and the line connecting the point $(c_0-1/k, g(c_0-1/k))$ on the graph of g and the point $(c_0,1)$ on the top edge of the box $[0,1]\times[0,1]$. In both cases $\tilde g_k$ satisfies the conditions (i)–(iii) for the concatenation operation $\vee$. Hence, if we let

$$p_k(x)=\begin{cases}\bigvee_{i=1}^{\infty}\tilde g_k(x), & 0<x\leqslant c_0,\\ 0, & x=0 \text{ or } c_0<x\leqslant 1,\end{cases}$$

then p_k is continuous at $x=0$ by Lemma 1(5). Moreover, $p_k\in X_0$ and either $\mathcal{R}^k[p_k]=p_k$ or $\mathcal{R}^{k+1}[p_k]=p_k$ by Lemma 1(3). Thus, p_k is a periodic point of $\mathcal{R}$. Since p_k and g can differ only on $[0,c_{-k}]$ and $[c_0-1/k,c_0]$, we have

$$\|p_k-g\|=O\left(\max\left\{\frac{1}{k},c_{-k}\right\}\right)\to 0\quad\text{as } k\to\infty.$$

This proves the proposition. □

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

Proposition 5. *The closure of the stable set of any $h \in X_0$ is X_0. More specifically, for any pair $g \in X_0, h \in X_0$ and any $\epsilon > 0$, there is an element $f_{h,g} \in X_0$ from the ϵ-neighborhood of g, i.e., $\| f_{h,g} - g\| < \epsilon$, so that $\mathcal{R}^n[f_{h,g}] = \mathcal{R}[h]$ for some $n \geqslant 0$.*

Proof. Let $g \in X_0$ and $h \in X_0$. As in the proof of Proposition 4 above, $c_{-k} = g^{-k}(c_0^g) \to 0$ as $k \to \infty$. Let $g_k = g|_{[c_{-k-1},1]}$. Then g_k and $\mathcal{R}[h]$ satisfy the conditions for the concatenation operation $\vee$, and we define $f_{h,g} = g_k \vee \mathcal{R}[h]$. Clearly $f_{h,g} \in X_0$. Since $f_{h,g}$ and g differ only possibly on $[0, c_{-k-1}]$, we have $\| f_{h,g} - g\| = O(c_{-k-1})$. Also, from the proof of Lemma 1(3) for $\vee$, either $\mathcal{R}^k[f_{h,g}] = \mathcal{R}[h]$ or $\mathcal{R}^{k+1}[f_{h,g}] = \mathcal{R}[h]$ depending on whether or not $c_{-k}c_0^h = c_{-k-1}$. Thus, for any $\epsilon > 0$, there is an integer $n \geqslant 1$ so that $\| f_{h,g} - g\| < \epsilon$ and $\mathcal{R}^n[f_{h,g}] = \mathcal{R}[h]$. □

Proposition 6. *$\mathcal{R}$ has the property of sensitive dependence on initial conditions. That is, there is a constant $\delta_0 > 0$ such that for any $g \in X_0$ and any small $\epsilon > 0$, there is an $h \in X_0$ and $n > 0$ satisfying $\|h - g\| < \epsilon$ and $\|\mathcal{R}^n[h] - \mathcal{R}^n[g]\| \geqslant \delta_0$.*

Proof. We need to construct an $h \in X_0$ for each $g \in X_0$ that satisfies the stated properties. To this end, we first demonstrate that any $g \in X_0$ can be properly separated from some element $\ell \in X_0$ by construction. More specifically, let $c_0 \in [0, 1]$ be the point of discontinuity of g. Then there is always a point in $(0, 1)$ denoted by c_0^ℓ that is no less than $1/4$ apart from c_0:

$$\left|c_0^\ell - c_0\right| \geqslant \frac{1}{4}.$$

Let ℓ be the line through the origin $(0, 0)$ and $(c_0^\ell, 1)$ over $[0, c_0^\ell]$ and 0 over $(c_0^\ell, 1]$. Then $\|\ell - g\|$ must be greater than the area of the trapezoid below the diagonal and over the interval $[c_0^\ell, c_0]$ if $c_0^\ell < c_0$ and $[c_0, c_0^\ell]$ if $c_0 < c_0^\ell$. This area is in turn greater than the area of the equal lateral right triangle which is the top part of the trapezoid. Since the area of that triangle is

$$\frac{1}{2}\left|c_0^\ell - c_0\right|\left|c_0^\ell - c_0\right| \geqslant \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32} := \delta_0,$$

it follows that

$$\|\ell - g\| > \delta_0 = \frac{1}{32}.$$

We are now ready to show the property of sensitive dependence on initial conditions. For each $g \in X_0$, we have $c_{-k} = g^{-k}(c_0) \to 0$ as $k \to \infty$. For each k, let ℓ_k be such a function associated with $\mathcal{R}^{k+1}[g]$ that is separated from $\mathcal{R}^{k+1}[g]$ by at least δ_0 amount. Moreover, we impose the condition that $c_0^{\ell_k} c_{-k} \neq c_{-k-1}$ in the construction of ℓ_k. Let $g_k = g|_{[c_{-k-1},1]}$ and define

$$h_k = g_k \vee \ell_k.$$

It is obvious that $\|h_k - g\| \leqslant c_{-k-1} \to 0$ since h_k and g differ only in the interval $[0, c_{-k-1}]$ with $h_k|_{[c_{-k-1},1]} = g_k = g|_{[c_{-k-1},1]}$. However, because $c_0^{\ell_k} c_{-k} \neq c_{-k-1}$, we have by the definition of $\vee$ that

$$\left\|\mathcal{R}^{k+1}[h_k] - \mathcal{R}^{k+1}[g]\right\| = \left\|\ell_k - \mathcal{R}^{k+1}[g]\right\| > \delta_0. \qquad \square$$

Proposition 7. *There are infinitely many dense orbits in X_0.*

Proof. The proof is based on the fact that the $L^1[0,1]$ space is separable, i.e., having a countable dense set [12]. To be precise, let $\mathcal{D}_1$ denote the subset of $L^1[0,1]$ that contains piecewise continuous and piecewise linear functions connecting vertexes of rational coordinates, in particular, with vertexes having the x-coordinates in the form of i/n for $0 \leqslant i \leqslant n$ and $n \geqslant 2$. Clearly $\mathcal{D}_1$ is countable and dense. For each $g \in X_0$, we can certainly approximate it by a sequence of functions g_n from $\mathcal{D}_1$ each of which is (i) continuously increasing over $[0, c_0^{g_n}]$, i.e., $g_n(x_1) < g_n(x_2), 0 \leqslant x_1 < x_2 \leqslant c_0^{g_n}$; (ii) above the diagonal $y = x$ over $[0, c_0^{g_n}]$, i.e., $g_n(x) > x, 0 < x \leqslant c_0^{g_n}$; (iii) vanishing at 0 and in $(c_0^{g_n}, 1]$. In other words, such a sequence can come from X_0. That is, X_0 itself is separable with the countable dense set $\mathcal{D}_2 = X_0 \cap \mathcal{D}_1$. Next for each $g \in \mathcal{D}_2$ we modify it to get a sequence by taking the following two steps. (1) If $g(c_0^g) = 1$, we do nothing about the discontinuity point c_0^g and set $g_n = g$. (2) Otherwise, $g(c_0^g) < 1$. Then we construct a sequence g_n with (i) $c_0^{g_n} = c_0^g + 1/n$; (ii) $g_n(x) = g(x)$ for $0 \leqslant x \leqslant c_0^g$ and $c_0^g + 1/n < x \leqslant 1$; (iii) g_n is the line connecting $(c_0^g, g(c_0^g))$ and $(c_0^{g_n}, 1)$. It is trivial to see that $g_n \in \mathcal{D}_2$ and $g_n \to g$ in L^1. That is, g_n is everything of any other $\mathcal{D}_2$ elements except that $g_n(c_0^{g_n}) = 1$. Denote this subset of $\mathcal{D}_2$ by $\mathcal{D}_3 \subset \mathcal{D}_2 \subset X_0$. Then we know $\mathcal{D}_3$ is countable and dense in $\mathcal{D}_2$, so is dense in X_0. We further modify $\mathcal{D}_3$ as follows. For each $g \in \mathcal{D}_3$, we have $c_{-k} = g^{-k}(c_0) \to 0$ as $k \to \infty$ as $x = 0$ is the only fixed point of g. We construct a sequence h_k each is the function g restricted on $[c_{-k}, c_0]$, i.e., $h_k = g|_{[c_{-k}, c_0]}$. This sequence $\{h_k\}$ has the property that by making any L^1-extension of h_k to the left-over interval $[0, c_{-k}]$, we will always have

$$\|h_k - g\|_{L^1[0,c_0]} \leqslant c_{-k} \to 0 \quad \text{as } k \to \infty.$$

Denote by $\mathcal{D}_4$ the set of all $h_k = g|_{[c_{-k}, c_0]}$ for all $g \in \mathcal{D}_3$. Then $\mathcal{D}_4$ is a countable set. Also, although $\mathcal{D}_4$ is not a subset of X_0, it can be treated to be dense in X_0 because for each $g \in X_0$ there is a sequence $\{g_{n_k}\}$ from $\mathcal{D}_4$ such that with an arbitrary extension to $[0, a_k]$ and 0 to $[c_0^{g_{n_k}}, 1]$ for each g_{n_k} with $[a_k, c_0^{g_{n_k}}]$ the domain of g_{n_k}, we have $g_{n_k} \to g$ as $k \to \infty$.

We are now ready to construct a dense orbit in X_0. Since $\mathcal{D}_4$ is countable, we have

$$\mathcal{D}_4 = \{g_1, g_2, \ldots\}.$$

We now construct

$$g_* = \begin{cases} \bigvee_{i=1}^{\infty} g_i(x), & 0 < x \leqslant c_0^{g_1}, \\ 0, & x = 0 \text{ or } c_0^{g_1} < x \leqslant 1. \end{cases}$$

It is obvious that by the definition of the concatenation operation $\vee$, g_* is continuous and increasing over the left open interval $(0, c_0^{g_1})$. Let $c_0 = c_0^{g_*}$ and $c_{-k} = g_*^{-k}(c_0)$. It is obvious that $c_{-k} \in [0, c_0)$ exist for all $k \geqslant 0$ by the definition of infinite concatenation as in $\bigvee_{i=1}^{\infty} g_i$. Therefore, $c_{-k} \searrow x_*$ exists as $k \to \infty$ and $x_* \in [0, c_0)$ is a fixed point of g_*. To show g_* is continuous at $x = 0$ and $g_* \in X_0$, we only need to show that $x_* = 0$. Suppose otherwise that $x_* > 0$. Then, $g_0 = \lim_{k\to\infty} \mathcal{R}^k[g_*]$ must exist and g_0 is a fixed point of $\mathcal{R}$ with $g_0(1^-) = 1$. On the other hand, by the definition of $\vee$ we have $g_n(x) = \mathcal{R}^k[g_*](x)$ for some $k \geqslant 0$, with k depending on n, and for all x from g_n's domain of definition $[a_n, c_0^{g_n}]$. The existence of the limit $g_0 = \lim_{k\to\infty} \mathcal{R}^k[g_*]$ forces the conclusion that $c_0^{g_n} \to 1$ as $n \to \infty$. Since $\mathcal{D}_4 = \{g_n\}$ can be regarded as a dense set of X_0, the existence of the limit $c_0^{g_n} \to 1$ would imply that every element $g \in X_0$ must have the property that $g(1^-) = 1$. This is certainly a contradiction to the fact that $x = 0$ is the only fixed point for every element $g \in X_0$. This completes the proof that $g_* \in X_0$.

We are now ready to show that the orbit through g_* is dense in X_0. In fact, for any $g \in X_0$ and any $\epsilon > 0$, there is a $g_k \in \mathcal{D}_4$ that is ϵ-close to g with any arbitrary X_0-extension of g_k to the left of its domain and 0 extension to the right of its domain. By the definition of the concatenation operation $\vee$ there is an integer n such that $\mathcal{R}^n[g_*] = g_k \vee \bigvee_{i=k+1}^{\infty} g_i$ over $[0, c_0^{g_k}]$ and 0 over $[c_0^{g_k}, 1]$. Hence, $\mathcal{R}^n[g_*]$ is ϵ-close to g. This shows that the orbit $\{\mathcal{R}^n[g_*]\}$ is dense in X_0.

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

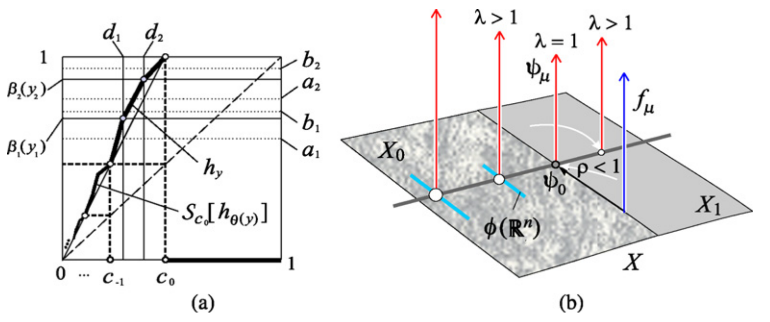

Fig. 3. (a) A schematic illustration for conjugating a 2-dimensional map $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to a sub-dynamics of $\mathcal{R}$. (b) A depiction for the dynamics of $\mathcal{R}$.

Last, from the construction above we clearly see there are infinitely many ways to construct such dense orbits. □

Proposition 8. *Any finite dimensional mapping is conjugate to $\mathcal{R}$ on a subset of X_0 and there are infinitely many such subsets of X_0. More precisely, for any finite dimensional mapping $\theta : \mathbb{R}^n \to \mathbb{R}^n$, there are infinitely many conjugate mappings $\phi : \mathbb{R}^n \to X_0$ such that $\mathcal{R} \circ \phi = \phi \circ \theta$.*

Proof. We need to construct a conjugacy ϕ for each mapping $\theta : \mathbb{R}^n \to \mathbb{R}^n$ that maps any point $y \in \mathbb{R}^n$ to a corresponding element $\phi(y) \in X_0$ so that $\mathcal{R} \circ \phi(y) = \phi \circ \theta(y)$. The construction to be used below will show that there are infinitely many such ϕ for every mapping θ.

We start by fixing any $\lambda > 1$ and the ray r_λ considered in the proof of Proposition 3 above. Here $r_\lambda(x) = \lambda x, 0 \leqslant x < 1/\lambda$, and $r_\lambda(x) = 0, 1/\lambda < x \leqslant 1$. The point of discontinuity is $c_0 = 1/\lambda$ and $c_{-k} = 1/\lambda^{k+1}$ with $r^{-1}(c_{-k}) = c_{-k-1}$, $k = 0, 1, 2, \ldots$. The goal is to construct for each $y \in \mathbb{R}^n$ an element $g = \phi(y) \in X_0$ with the property that $c^g_{-k} = c_{-k} = 1/\lambda^{k+1}$, $k = 0, 1, 2, \ldots$, and $\mathcal{R} \circ \phi(y) = \mathcal{R}[g](y) = \phi \circ \theta(y)$, see Fig. 3(a). In fact, we will construct g to be a piecewise linear curve from X_0 having exactly $n + 1$ line segments over each interval $[c_{-k-1}, c_{-k}]$, $k = 0, 1, 2, \ldots$. The key step is in constructing the piece over the first interval $[c_{-1}, c_0]$ by embedding $\mathbb{R}^n$ into the space of piecewise linear functions from $[c_{-1}, c_0]$ to $[c_0, 1]$.

To this end, we first arbitrarily pick and fix n points $c_{-1} < d_1 < d_2 < \cdots < d_n < c_0$. Denote the images of d_i under r_λ by $a_i = r_\lambda(d_i) = \lambda d_i$, $1 \leqslant i \leqslant n$. By r_λ's monotonicity, this gives $c_0 = r_\lambda(c_{-1}) < a_1 < a_2 < \cdots < a_n < 1 = r_\lambda(c_0)$. We then arbitrarily pick and fix b_i so that $a_i < b_i < a_{i+1}$, $i = 1, 2, \ldots, n$, with $a_{n+1} = 1$. We are now ready to embed $\mathbb{R}^n$ into the space of piecewise linear functions from $[c_{-1}, c_0]$ to $[c_0, 1]$. More specifically, let $\beta_i : \mathbb{R} \to (a_i, b_i)$ be any 1-to-1 and onto map and let $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with y_i being the ith coordinate. We then define h_y to be the graph over interval $[c_{-1}, c_0]$ that connects the vertex points $(c_{-1}, c_0), (d_1, \beta_1(y_1)), \ldots, (d_i, \beta_i(y_i)), \ldots, (d_n, \beta_n(y_n))$, and $(c_0, 1)$ with line segments. Because of the choice that $\beta_i(y_i) < b_i < a_{i+1} < \beta_{i+1}(y_{i+1})$, each line through $(d_i, \beta_i(y_i))$ and $(d_{i+1}, \beta_{i+1}(y_{i+1}))$ must be increasing. Hence h_y is increasing in $[c_{-1}, c_0]$. It is continuous by construction and $h_y(c_{-1}) = c_0$, $h_y(c_0) = 1$. It lies above the diagonal because it lies above the ray r_λ. Therefore $h_y \vee h_z$ is well defined with any $y, z \in \mathbb{R}^n$, in particular, with $z = \theta(y)$. We now complete our construction for $g = \phi(y)$ by defining

$$g = \phi(y) = \begin{cases} \bigvee_{i=0}^{\infty} h_{\theta^i(y)}(x), & 0 < x \leqslant c_0, \\ 0, & x = 0 \text{ or } c_0 < x \leqslant 1, \end{cases}$$

see Fig. 3(a). By the definition of $\vee$ we have $\mathcal{R}[\phi(y)](x) = \bigvee_{i=1}^{\infty} h_{\theta^i(y)}(x) = \phi(\theta(y))(x)$ for $x \in [0, c_0)$ and 0 otherwise. That is, $\mathcal{R}[\phi(y)] = \phi(\theta(y))$ as desired. Finally, we point out that there are infinitely many ways to construct the conjugacy ϕ above by, e.g., starting with distinct rays r_λ for $\lambda > 1$, or by varying the parameters d_i, a_i, b_i. (Also, it is easy to see by the construction above that this result can be generalized to include mappings on product spaces, e.g., $\mathbb{R}^\omega$, which include shift maps.)

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004

Last the construction above shows that the finite dimensional system θ need not to be deterministic. It can be a purely stochastic process for which all (forward) temporal sequences of the system's probabilistic outcomes are used instead to conjugate orbits of $\mathcal{R}$ in the subset $\phi(\mathbb{R}^n)$. □

4. Closing remarks

Fig. 3(b) is a dynamical representation of the renormalization, encapsulating all results of this paper and the previous one in the series. The grainy texture for the chaotic space X_0 is meant to represent the operator's dense orbits. The one-parameter, spike renormalizable family f_μ is a qualitative representation of a typical family of the Poincaré return maps from our circuit models of neurons, satisfying the universality conditions of [5]. It is transversal to the center-stable manifold of the identity map ψ_0, arising from the boundary between the chaotic set X_0 and the non-chaotic one X_1. The family ψ_μ represents the 1-dimensional, eigenvalue-1 but weakly-expanding center-unstable manifold of ψ_0. The convergence of f_μ to ψ_μ under the iteration of $\mathcal{R}$ gives rise to the universality of the isospiking bifurcation sequence $\mu_n \sim \frac{1}{n}$.

We end the paper by pointing out that there are many interesting and nontrivial topological properties of the embedding maps that are open for future studies. For example, the continuity or differentiability of the embedding ϕ from $\mathbb{R}^n$ to X_0 at a θ depends on the continuity or differentiability of θ from $\mathbb{R}^n$ to itself. Also, a preliminary investigation seems to suggest that the embedding map will preserve the Lyapunov exponents of deterministic dynamical systems if the exponents are no greater than the expanding rate λ (or λ^2) of the fixed point r_λ near which the conjugating embedding takes place.

References

[1] T.R. Chay, J. Keizer, Minimal model for membrane oscillations in the pancreatic beta-cell, Biophys. J. 42 (1983) 181–190.
[2] B. Deng, Conceptual and circuit models of neurons, J. Integrative Neurosci. 8 (2009) 255–297.
[3] B. Deng, Metastability and plasticity in some conceptual models of neurons, J. Integrative Neurosci. 9 (2010) 31–47.
[4] B. Deng, Optimizing a communication system by neural circuits: the magic number 4 and golden ratio, 2007.
[5] B. Deng, Neural spike renormalization. Part I — Universal number 1, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.002, in press.
[6] M. Feigenbaum, Quantitative universality for a class of nonlinear transformations, J. Stat. Phys. 19 (1979) 25–52.
[7] M. Feigenbaum, The universal metric properties of nonlinear transformation, J. Stat. Phys. 21 (1979) 669–709.
[8] J.L. Hindmarsh, R.M. Rose, A model of neuronal bursting using three coupled first order differential equations, Proc. R. Soc. Lond. Ser. B 221 (1984) 87–102.
[9] A.L. Hodgkin, A.F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, J. Physiol. 117 (1952) 500–544.
[10] C. Morris, H. Lecar, Voltage oscillations in the barnacle giant muscle fiber, Biophys. J. 35 (1981) 193–213.
[11] J. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 59 (1964) 2061–2070.
[12] W. Rudin, Principles of Mathematical Analysis, McGraw–Hill Publishing Company, 1953.

Please cite this article in press as: B. Deng, Neural spike renormalization. Part II — Multiversal chaos, J. Differential Equations (2010), doi:10.1016/j.jde.2010.10.004