On Stability of Heegaard Splittings
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December 3, 2018
Outline

Background and definitions
- 2-Manifolds and 3-Manifolds
- Compression Body
  - Handlebody
  - Trivial Compression Body
- Heegaard Splitting
- Stabilization

Stabilization Theorem
- Proof Idea
2-Manifolds

Def: Topological spaces which are locally homeomorphic to $\mathbb{R}^2$. 
3-Manifolds

Def: Topological spaces which are locally homeomorphic to $\mathbb{R}^3$. 
Compression bodies
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Call the “outside” surface $\partial_+ C$ and let $\partial_- C := \partial C \setminus \partial_+ C$. 

Dual construction of compression bodies
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Here we start with $\partial_- C$ and build up to $\partial_+ C$ rather than the other way around.
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Compression bodies
Types of compression bodies

Def: A **handlebody** is a compression body $C$ where $\partial_- C = \emptyset$.

Def: A **trivial compression body** is $S \times I$ for some surface $S$. 
Remark: All handlebodies retract to a graph known as the **spine** of the handlebody.
Heegaard splitting

Def: A **Heegaard splitting** of a 3-manifold $M$ is a pair of compression bodies $V$ and $W$ such that

- $V \cap W = \partial_+ V = \partial_+ W = F$
- $V \cup W = M$.

This is denoted $(F, V, W), (M, F)$, or $V \cup_F W$. 
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Stably equivalent

Def: Two Heegaard splittings \((M, F)\) and \((M, F')\) of the same 3-manifold \(M\) are **stably equivalent** if there exists some Heegaard splitting \((M, F'')\) such that:
Stabilization Theorem

**Theorem (Reidemeister, Singer)**

*Any two Heegaard splittings of an orientable, closed 3-manifold are stably equivalent.*

Proof Idea/Beginnings: Let \((F, V, W)\) and \((F', V', W')\) be Heegaard splittings of an orientable, closed 3-manifold \(M\). Isotopy \(V'\) and \(W\) to be disjoint.
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Let \(X = V \setminus V' = W' \setminus W\).
Let \((S, Y, Y')\) be a Heegaard splitting of \(X\).
Reference

Lei Fengchun
On Stability of Heegaard Splittings
(1999)