Polynomials and recurrence relations

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1. (Putnam, 1986) What is the units digit of $\lfloor \frac{10^{20000}}{10^{100}+3} \rfloor$? Here $\lfloor x \rfloor$ is the floor function, that is $\lfloor x \rfloor$ is the largest integer $\leq x$.

2. (Putnam, 1995) Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$$

3. (Putnam, 1988) Prove that there exists a unique function $f : \mathbf{R}^+ \to \mathbf{R}^+$ ($\mathbf{R}^+$ is the set of positive reals) such that

$$f(f(x)) = 6x - f(x) \text{ and } f(x) > 0 \text{ for all } x > 0.$$

4. (Putnam, 1996) Define a *selfish* set to be a set which has its own cardinality as an element. Find the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is selfish sets none of whose proper subsets are selfish.

5. (Putnam, 1999) Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = f_1(x)^2 + \dots + f_k(x)^2.$$

6. (Putnam, 1992) For nonnegative integers n and k define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.$$

7. (Putnam, 1985) Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$ and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1), n \geq 0.$$

Find the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Hint: Find $f_n(x)$ first.