Number Theory

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1. Show that no positive integers x, y, z satisfy $x^2 + 10y^2 = 3z^2$.

2. (First W.L Putnam competition, 1939) Prove that for no integer $n > 1$ does n divide $2^n - 1$.

3. (Austrian-Polish math competition, 1999) Solve in positive integers the equation $x^{x+y} = y^{y-x}$.

4. (Romanian Mathematical Olympiad, 1997) Let $A = \{a^2 + 2b^2 : a, b \in \mathbf{Z}, b \neq 0\}$. Show that if p is a prime such that $p^2 \in A$ then $p \in A$.

5. (Putnam, 1994) Find all positive integers that are within 250 of exactly 15 perfect squares.

6. (Putnam, 1997) Let N_n denote the number of ordered n-tuples of positive integers $(a_0, a_1, \ldots, a_n)$ such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1$. Determine whether N_{10} is even or odd.

7. (Putnam, 1991) Let p be an odd prime and let $\mathbf{Z}_p$ denote the integers modulo p. How many integers are in the set $\{x^2 : x \in \mathbf{Z}_p\} \bigcap \{y^2+1 : y \in \mathbf{Z}_p\}$?

8. (Putnam, 1985) Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$. Which integers between 0 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i.

9. (Putnam, 1991) Let p be an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 (\text{mod } p^2).$$

10. (Putnam, 1994) For any integer a set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d (\text{mod } 10100)$ implies $\{a, b\} = \{c, d\}$.

The following information might be useful for these problems:

- If a, b, n are integers we say $a \equiv b(\text{mod } n)$ if $a - b$ is divisible by n.
- If p is a prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1(\text{mod } p)$. (Fermat's Little Theorem)
- If n, a are positive integers and $gcd(n, a) = 1$ then $a^{\phi(n)} \equiv 1(\text{mod } n)$ where $\phi(n)$ is is defined by

  $$\phi(p_1^{n_1} p_2^{n_2} \ldots p_k^{a_k}) = p_1^{n_1-1}(p_1 - 1)p_2^{n_2-1}(p_2 - 1) \ldots p_k^{n_k-1}(p_k - 1).$$

  (Euler's Theorem)
- If p is a prime , then $(p - 1)! \equiv -1(\text{mod } p)$. (Wilson's Theorem)
- The set of integers mod n is $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$. When working with elements of $\mathbf{Z}_n$, addition and multiplication are defined mod n, that is the result of $a + b$ in $\mathbf{Z}_n$ is the remainder of $a + b$ when divided by n and the result of $a \cdot b$ in $\mathbf{Z}_n$ is the remainder of $a \cdot b$ when divided by n.