Combinatorics

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1. (-) (Putnam, 1985) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(a) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and

(b) $A_1 \cup A_2 \cup A_3 = \emptyset$, where $\emptyset$ is the empty set.

Express the answer in the form $2^a 3^b 5^c 7^d$ where a, b, c, d are nonnegative integers.

2. How many integers less than 1000 are not divisible by 2, 3, or 5?

3. * n hat-wearing men go to a party. (Their hats are all different.) They check in their hats upon arrival and receive a randomly selected hat back at departure. How many ways are there for hats to be returned such that **none** of the men receives his own hat back?

4. * Given a graph with n vertices, prove that either it contains a triangle, or there exists a vertex that is the endpoint of at most $\lfloor n/2 \rfloor$ edges.

5. (-) Consider a party with 6 people. For any two of these people they've either met before (acquaintances) or not (strangers). Prove that there are either 3 people who are mutual acquaintances or 3 people who are mutual strangers.

6. Show that if every point in the plane is colored either black or white, then there exists an equilateral triangle whose vertices are colored by the same color.

7. (Putnam, 1996) Suppose that each of 20 students has made a choice for anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

8. * Prove the following identity by counting a certain set in two different ways.

$$\sum_{k=l}^{n} \binom{k}{l}\binom{n}{k} = \binom{n}{l} 2^{n-l}.$$

9. ***(Putnam, 1999)** Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

10. In a society of n people, any two persons who do not know each other have exactly two common acquaintances, and any two persons who know each other dont have other common acquaintances. Prove that in this society every person has the same number of acquaintances.