Math 918 **Computer Worksheet 4**

1. (Borel-fixed monomial ideals) Let $R = \mathbb{Q}[x_1, \ldots, x_n]$ and $u = x_1 x_2 \cdots x_n$. Consider the set of monomials that can be obtained from u by repeatedly replacing a variable x_i by another variable x_j such that $j < i$. The ideal generated by all of these monomials is called the *principal Borel ideal* generated by $x_1 x_2 \cdots x_n$ and denoted $Borel(x_1 x_2 \cdots x_n)$.

(a) Compute the principal Borel ideal generated by u for several values of n. Here is an example for $n = 3$:

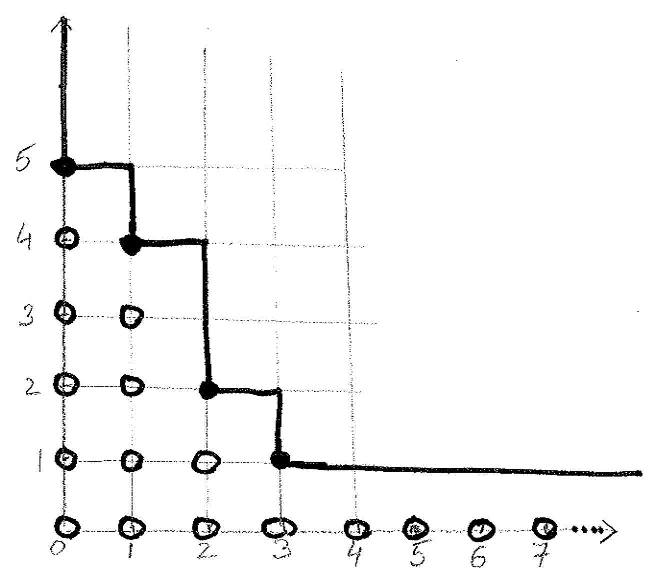
```
R=QQ[x_1..x_3];
u=product gens R;
I=ideal borel matrix {{u}}
```

(b) Find the number of minimal generators for $Borel(x_1 \cdots x_n)$ for several values of n. `Try to do this using a for loop.`

(c) Can you recognize the sequence of numbers in part (b)? Conjecture a formula for the number of minimal generators for $Borel(x_1 \cdots x_n)$. Can you prove it ?

2. (An exploration for monomial ideals in 2 variables - adapted from [MS])) Let $R = \mathbb{Q}[x, y]$ and consider the ideal $I = (y^5, xy^4, x^2y^2, x^3y)$ in R.

(a) Remind yourselves how the staircase diagram below relates to I.



(b) Compute (by hand) the sum of all monomials *outside* of I. Your task is to express this sum as a rational function with denominator $(1 - x)(1 - y)$. This is called the *multigraded Hilbert function* of R/I.

(c) Recompute the sum above using *inclusion-exclusion.* Start with the sum of all monomials in R (you can express this as a very easy rational function). Then for each minimal generator m_i subtract off the sum of the monomials in the principal ideal generated by m_i (i.e the monomials in the positive orthant starting at m_i). Now add back in what you have subtracted twice.

(d) Now let's check your answer using *Macaulay2*:

```
R=QQ[x,y,Degrees=>{{1,0},{0,1}},Heft=>{1,1}];
I=monomialIdeal(y^5, x*y^4, x^2*y^2, x^3*y);
HS= hilbertSeries I;
use degreesRing R;
substitute(HS,{T_0=>x,T_1=>y})
```

(e) Can you relate the numerator of the Hilbert series above back to the staircase diagram?

(f) Compute the minimal free resolution of I:

```
(res I).dd
```

Can you relate the *minimal* syzygies on the generators of I back to the staircase diagram?

3. (An exploration for monomial ideals in 3 variables - adapted from [MS]) Let $R = QQ[x, y, z]$ and consider the ideal $I = (x^3y^2z, xy^3z^2, x^2yz^3, x^4, y^4, z^4)$ in R.

(a) Look at the staircase diagram below (on left). What does it represent? How do the black (inner) corners relate to I?

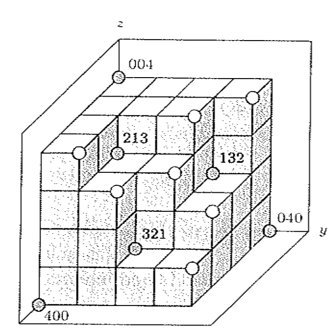
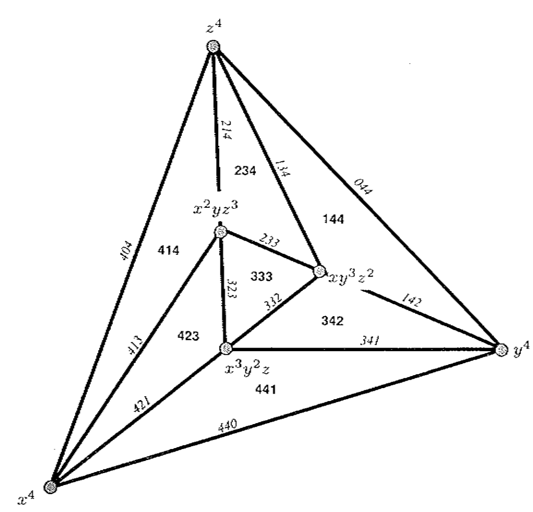


Figure 3.1: A staircase diagram in three variables



(b) Look at the graph above (on the right). What appears to be the labeling rule for the edges and faces of this planar graph? How do the black and white dots in the previous picture relate to this graph?

(c) First, let's compute the sum of all the monomials in I. To do this, we'll compute the numerator of the *multigraded Hilbert series* of R/I (we'll find out what this is in class later) as follows:

```
R=QQ[x,y,z,Degrees=>{{1,0,0},{0,1,0},{0,0,1}},Heft=>{1,1,1}];
I=ideal(x^3*y^2*z,x*y^3*z^2,x^2*y*z^3,x^4,y^4,z^4);
HS= hilbertSeries I;
use degreesRing R;
substitute(HS,{T_0=>x,T_1=>y,T_2=>z})
```

Interpret the result above in terms of the graph in part (b).

(d) Compute the minimal free resolution of I using the command

```
(res I).dd
```

Interpret the maps in the resolution in terms of the graph above.

(e) Next, let's *polarize* I, that is in each generator of I we'll replace the n^{th} power of x by $x_1 \dots x_n$ and similarly for y and z (this yields a *squarefree* monomial ideal). How are the minimal free resolutions of these two ideals related? can you give a theoretical argument why this is the case?

```
x=symbol x; y=symbol y; z=symbol z;
S = ZZ[x_1..x_4, y_1..y_4, z_1..z_4];
Ipol=monomialIdeal(x_1*x_2*x_3*y_1*y_2*z_1,x_1*y_1*y_2*y_3*z_1*z_2,
x_1*x_2*y_1*z_1*z_2*z_3,x_1*x_2*x_3*x_4,y_1*y_2*y_3*y_4,z_1*z_2*z_3*z_4)
(res Ipol).dd
(res I).dd
```

References

[MS] E. Miller and B. Sturmfels *Combinatorial Commutative Algebra*, Springer 2005