Math 918 — Computer Worksheet 3

1. (The four bar kinematic problem) Let $A = (0,0), B = (0,2), C = (x_1, y_1)$ and $D = (x_2, y_2)$ in $\mathbb{A}^2$. Each of the bars in the picture below are labeled with their respective length. They are connected together by completely flexible joints, which can rotate fully and flip over. The points A and B are fixed but C, D and E are allowed to move subject to the obvious restrictions imposed by the rigidity of the bars. It turns out that if one allows the points C, D and E to move to all possible positions, the set of points that E can move to traces a curve in $\mathbb{A}^2$.

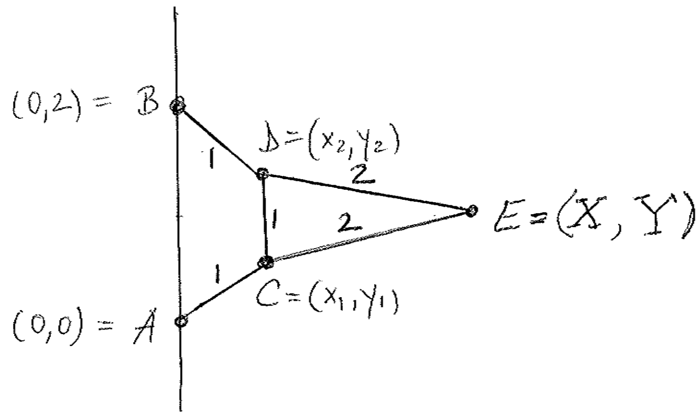
(a) Set up an ideal in a polynomial ring that models the restrictions in the problem.

(b) Find the equations of the curve $\mathcal{C}$ traced by E (i.e. find $\mathbb{I}(\mathcal{C})$).

(c) How many connected components does $\mathcal{C}$ have (in $\mathbb{A}^2_{\mathbb{R}}$)?

Hint: you will likely use elimination theory for this problem. Look up the

`selectInSubring` *command or the* `MonomialOrder => Eliminate n` *option.*



2. (Homogenization) Define the coordinate ring $R = \mathbb{R}[w,x,y,z]$ of $\mathbb{A}^4$ and the ideal $I = (w^2 - x, w^3 - y, w^4 - z)$ in R. Define the new ring $S = k[w,x,z,y,h]$.

(a) Let J be the ideal in S obtained from I by homogenizing the given generators of I. This ideal can be gotten with the command `homogenize`.

(b) Saturate the ideal J with respect to the homogenizing variable h (i.e either use the command `saturate` or take colons $J : (h^n)$ for $n \geq 1$ until the colon stabilizes). Call the saturated ideal you obtained L. We call $\mathbb{V}(L)$ the *projective closure* of $\mathbb{V}(I)$.

(c) Compute the Gröbner basis for I with respect to GrLex. Use the command

```
flatten entries gens gb I
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for a list of the elements in the Gröbner basis. Homogenize the elements in this Gröbner basis and call the ideal generated by these homogeneous polynomials L'.

(d) Compare L and L'. What do you notice?

3. (Linkage) The *projective twisted cubic* is the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(s:t) \mapsto (s^3 : s^2t : st^2 : t^3).$$

(a) Let $R = \mathbb{Q}[x_0 \ldots x_6]$ be the coordinate ring of $\mathbb{P}^3$. Find the ideal I of R defining the *projective twisted cubic*. Can you describe this ideal in a different way?

(b) Let C be the ideal generated by any two of the minimal generators of I. Compute $C : I$. What kind of projective variety is $\mathbb{V}(C : I)$?

(c) What is the union of $\mathbb{V}(I)$ and $\mathbb{V}(C : I)$?

(d) How to the *degrees* (or *multiplicities*) of $I, C : I$ and C relate ? You may use the command `degree` to find them.

Here, I am using the algebraic geometric notion of degree which means, roughly, the number of intersection points of a variety with a general enough hyperplane.

(e) Below we have a grid of lines in $\mathbb{A}^2$ (or if you prefer, work in $\mathbb{P}^2$). Say the marked points are integer lattice points with the origin in the bottom left corner.

Write down the ideal I defining the set of points marked with solid black dots. Find the ideal J defining the set of points marked with circles using the ideal C corresponding to the union of lines in the picture. How could you check your answer?

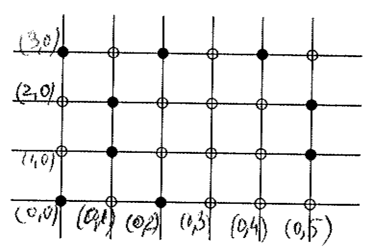

FIGURE 1. Geometric link of points