BOUNDING PROJECTIVE DIMENSION

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1. Introduction

The use of algorithms in algebra as well as the study of their complexity was initiated before the advent of modern computers. Hermann [25] studied the ideal membership problem, i.e determining whether a given polynomial is in a fixed homogeneous ideal, and found a doubly exponential bound on its computational complexity. Later Mayr and Meyer [31] found examples which show that her bound was nearly optimal. Their examples were further studied by Bayer and Stillman [3] and Koh [28] who showed that these ideals also had syzygies whose degrees are doubly exponential in the number of variables of the ambient ring.

This survey addresses a different measure of the complexity of an ideal, approaching the problem from the perspective of computing the minimal free resolution of the ideal. Among invariants of free resolutions, we focus on the projective dimension, which counts the number of steps one needs to undertake in finding a minimal resolution; the precise definition of projective dimension is given in Section 2. In this paper we discuss estimates on the projective dimension of cyclic graded modules over a polynomial ring in terms of the degrees of the minimal generators of the defining ideal. We also establish connections to another well-known invariant, namely regularity.

The investigation of this problem was initiated by Stillman who posed the following question:

Stillman's Question 1.1. [35, Problem 3.14] *Let R be any standard graded polynomial ring, suppose $I = (f_1, \ldots, f_n) \subset R$ is a homogeneous ideal and $f_1, \ldots, f_n$ is a minimal set of generators of I. Is there a bound on the projective dimension of R/I depending only on $d_1, \ldots, d_n$, where $d_i = \deg(f_i)$ for $i = 1, \ldots, n$?*

Note that the degrees d_i of the generators as well as the number n of generators of I are part of the data with which we may bound the projective dimension $\operatorname{pd}(R/I)$, however Stillman's Question asks for a bound independent of the number of variables.

To completely answer Stillman's Question one would ideally like to describe:

(1) a bound for $\operatorname{pd}(R/I)$ in terms of $d_1, \ldots, d_n$ which is always valid,
(2) examples of ideals I where the bound in (1) is the best possible,
(3) much better bounds for $\operatorname{pd}(R/I)$ valid if I satisfies special conditions.

In this survey we gather recent results which partially answer (2) and (3). We remark that question (1) is still wide open. We hope this paper serves as a convenient survey of these results and spurs future work in this area.

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In the next section, we fix notation for the remainder of the paper and explain the equivalence of Stillman's problem to the analogous problem on bounding Castelnuovo-Mumford regularity. In Section 3, we summarize the main results that explore special cases of Stillman's Question, including a sketch of the bound for ideals generated by three quadratics, three cubics and arbitrarily many quadratics. In Section 4, we present several examples of ideals with large projective dimension giving large lower bounds on possible answers to Stillman's Question. In Section 5, we summarize some related bounds on projective dimension that are distinct from Stillman's Question. We close in Section 6 with some questions and possible approaches to Stillman's Question.

2. Background and an Equivalent Problem

For the rest of this paper, we stick to the following conventions: We use $R = K[x_1, \ldots, x_N]$ to denote a polynomial ring over an arbitrary field K in N variables and we let $\mathfrak{m} = (x_1, x_2, \ldots, x_N)$ denote the graded maximal ideal. We consider R as a standard graded ring with $\deg(x_i) = 1$ for all $i = 1, \ldots, N$. We call a homogeneous polynomial a *form*. We denote by R_i the K-vector space of degree-i forms in R. Hence $R = \oplus_{i \geq 0} R_i$ as a K-vector space. We also denote by $R(-d)$ the rank one free module with generator in degree d so that $R(-d)_i = R_{i-d}$. Given any finitely generated R-module M, a *free resolution* $F_\bullet$ of M is an exact sequence of the form:

$$\mathbf{F}_\bullet : F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_{s-1}} F_{s-1} \xleftarrow{\partial_s} F_s \xleftarrow{\partial_{s+1}} \cdots$$

where F_i is a free module and $M = F_0/\operatorname{Im}(\partial_1)$. The length of a resolution $F_\bullet$ is the greatest integer n such that $F_n \neq 0$, if such an integer exists; otherwise the length is infinite. We then define the *projective dimension* of M, denoted $\operatorname{pd}(M)$, to be the minimum of the lengths of all free resolutions of M. When M is graded, we require that the free resolution of M be graded, ∂_i is a graded map for all i. Moreover, $\mathbf{F}_\bullet$ is called *minimal* if $\operatorname{Im}(\partial_i) \subset \mathfrak{m}F_{i-1}$ for $i \geq 1$. The minimal graded free resolution of M is unique up to isomorphism, and it follows that $\operatorname{pd}(M)$ is the length of any minimal graded free resolution of M.

The projective dimension can be thought of as a measure of how far M is from being a free module, since finitely generated modules with projective dimension 0 are free. We note that over R every finitely generated graded projective module is free. This explains why the length of a free resolution is called the projective dimension.

It was Hilbert [26] who first studied free resolutions associated to graded modules over a polynomial ring. His Syzygy Theorem shows that every graded module over a polynomial ring has a finite, graded free resolution. (See [14] for a proof).

Theorem 2.1 (Hilbert [26]). *Every finitely generated graded module M over the ring $K[x_1, \ldots, x_N]$ has a graded free resolution of length $\leq N$. Hence $\operatorname{pd}(M) \leq N$.*

In this survey, we shall consider the projective dimension of homogeneous ideals $I \subset R$, with the exception of Section 3.2, where the homogeneity assumption is not required. By convention, we study the projective dimension of cyclic modules $\operatorname{pd}(R/I)$ rather than that of ideals, noting that $\operatorname{pd}(R/I) = \operatorname{pd}(I) + 1$ for all ideals I. Hilbert's Syzygy Theorem shows that $\operatorname{pd}(R/I) \leq N$ for all ideals I. Even for ideals, this bound is tight. The graded maximal ideal $\mathfrak{m} = (x_1, \ldots, x_N)$ defines a cyclic module $K \cong R/\mathfrak{m}$ with $\mathrm{pd}(R/\mathfrak{m}) = N$. In fact, the Koszul complex on the variables $x_1, \ldots, x_N$ gives a minimal free resolution of $R/\mathfrak{m}$ of length N.

For a graded free resolution $\mathbf{F}_\bullet$ of M, we write $F_i = \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M)$ are invariants of M and are called the *Betti numbers* of M. We often record these in a matrix, called the *Betti table* of M. By convention, we write $\beta_{i,j}$ in column i and row $j - i$.

	0	1	2	$\cdots$	i	$\cdots$
0:	$\beta_{0,0}(M)$	$\beta_{1,1}(M)$	$\beta_{2,2}(M)$	$\cdots$	$\beta_{i,i}(M)$	$\cdots$
1:	$\beta_{0,1}(M)$	$\beta_{1,2}(M)$	$\beta_{2,3}(M)$	$\cdots$	$\beta_{i,i+1}(M)$	$\cdots$
2:	$\beta_{0,2}(M)$	$\beta_{1,3}(M)$	$\beta_{2,4}(M)$	$\cdots$	$\beta_{i,i+2}(M)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
j:	$\beta_{0,j}(M)$	$\beta_{1,j+1}(M)$	$\beta_{2,j+2}(M)$	$\cdots$	$\beta_{i,i+j}(M)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Another way to measure the complexity of M is to look at the degrees of the generators of the free modules F_i. We define the *Castelnuovo-Mumford regularity* of M (or just the regularity of M) to be

$$\mathrm{reg}(M) = \max\{j \mid \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}.$$

Hence $\mathrm{pd}(M)$ is the index of the last nonzero column and $\mathrm{reg}(M)$ is the index of the last nonzero row in the Betti table of M. Regularity has many connections with algebraic geometry and Hilbert functions, for which we refer the interested reader to [15] and [34]. In general, both $\mathrm{reg}(R/I)$ and $\mathrm{pd}(R/I)$ may depend on the characteristic of the base field K, even in the case where I is a monomial ideal. (See e.g. [34, Example 12.4].) A complete answer to Stillman's Question should be independent of the base field, although an answer in any characteristic would be very interesting.

Example 2.2. Let $R = K[w, x, y, z]$ and let $I = (w^2, x^2, wy + xz)$. Then R/I has minimal graded free resolution:

$$R \xleftarrow{\begin{pmatrix} w^2 & x^2 & wy+xz \end{pmatrix}} R(-2)^3 \xleftarrow{\begin{pmatrix} -x^2 & 0 & -wy-xz & -xy & -y^2 \\ w^2 & -wy-xz & 0 & -wz & z^2 \\ 0 & x^2 & w^2 & wx & wy-xz \end{pmatrix}} R(-4)^5 \longleftarrow$$

$$\xleftarrow{\begin{pmatrix} y & z & 0 & 0 \\ w & 0 & 0 & z \\ 0 & -x & -y & 0 \\ -x & w & z & -y \\ 0 & 0 & w & x \end{pmatrix}} R(-5)^4 \xleftarrow{\begin{pmatrix} -z \\ y \\ -x \\ w \end{pmatrix}} R(-6) \longleftarrow 0$$

The Betti Table of M is then

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	3	-	-	-
2:	-	-	5	4	1

Therefore, we have $\mathrm{pd}(R/I) = 4$ and $\mathrm{reg}(R/I) = 2$.

One could pose a similar question to Question 1.1 by replacing the words "projective dimension" with "regularity" and asking for a bound on $\operatorname{reg}(R/I)$ purely in terms of the degrees of the generators.

Question 2.3 ([35, Problem 3.15]). *Let R be any polynomial ring and suppose $I = (f_1, \ldots, f_n) \subset R$ is an ideal. Is there a bound on $\operatorname{reg}(R/I)$ depending only on $d_1, \ldots, d_n$, where $d_i = \deg(f_i)$ for $i = 1, \ldots, n$?*

One of the primary motivations for studying Question 1.1 is that it is equivalent to Question 2.3.

Theorem 2.4 (Caviglia [34, Theorem 29.5],[17, Section 1.3]). *Let K a field. Fix a sequence of natural numbers $d_1 \leq \ldots \leq d_n$. The following are equivalent:*

(1) There exists a function $B(n, d_1, \ldots, d_n)$ such that $\operatorname{pd}_R(R/I) \leq B(n, d_1, \ldots, d_n)$ if R is any polynomial ring over K and $I \subset R$ is any graded ideal with a minimal system of homogeneous generators of degrees $d_1 \leq \ldots \leq d_n$.

(2) There exists a function $C(n, d_1 \ldots d_n)$ such that $\operatorname{reg}_R(R/I) \leq C(n, d_1, \ldots, d_n)$ if R is any polynomial ring over K and $I \subset R$ is any graded ideal with a minimal system of homogeneous generators of degrees $d_1 \leq \ldots \leq d_n$.

We outline a proof of this result below. First we recall a related bound on regularity. Similar to the existence of a bound on projective dimension given by the Hilbert basis theorem, there is a doubly exponential bound for the regularity of an ideal I expressed in terms of the number of variables of the ambient ring and the maximal degree of a minimal generator of I. This bound can be deduced from work of Galligo [22] and Giusti [23] in characteristic zero, as was observed by Bayer and Mumford [2, Theorem 3.7]. It was later proved in all characteristics by Caviglia and Sbarra [10].

Theorem 2.5. *Let $R = K[x_1, \ldots, x_N]$ be a polynomial ring over a field K. Let I be a graded ideal in R and let r be the maximal degree of an element in a minimal system of homogeneous generators of I. Then $\operatorname{reg}(I) \leq (2r)^{2^{N-2}}$.*

We use this bound to prove Theorem 2.4.

Proof of Theorem 2.4. Let $R = K[x_1, \ldots, x_N]$ throughout. Assume (1) holds and fix an ideal I with minimal generators of degrees $d_1 \leq \ldots \leq d_n$. Let $p = \operatorname{pd}(R/I) \leq B(n, d_1, \ldots, d_n)$. By the Auslander-Buchsbaum theorem, $\operatorname{depth}(R/I) = N - p$, which means that one can find a regular sequence of linear forms $\ell_1, \ldots, \ell_{N-p}$ on R/I. If ℓ is a linear nonzero divisor on R/I, one obtains a short exact sequence of the form

$$0 \longrightarrow \frac{R}{I}(-1) \xrightarrow{\cdot \ell} \frac{R}{I} \longrightarrow \frac{R}{I + (\ell)} \longrightarrow 0.$$

The mapping cone construction now yields $\operatorname{reg} \frac{R}{I} = \operatorname{reg} \frac{R}{I+(\ell)}$ and by induction

$$\operatorname{reg} \frac{R}{I} = \operatorname{reg} \frac{R}{I + (\ell_1, \ldots, \ell_{N-p})}.$$

Set $\overline{R} = R/(\ell_1, \ldots, \ell_{N-p})$. Then $R/(I + (\ell_1, \ldots, \ell_{N-p})) = \overline{R}/I\overline{R}$ can be viewed as a quotient algebra of the polynomial ring $\overline{R}$. The ring $\overline{R}$ is isomorphic to a polynomial ring in p variables, hence by applying Theorem 2.5 an upper bound on $\operatorname{reg} R/I$ is $(2d_n)^{2^{p-2}}$. One may now set

$$C(n, d_1, \ldots, d_n) = (2d_n)^{2^{B(n,d_1,\ldots,d_n)-2}}.$$

Conversely assume (2) holds and fix an ideal $I \subset R$. Denote by $\operatorname{gin}(I)$ the generic initial ideal of I with respect to the degree reverse lexicographic ordering on the monomials of R. This term order has good properties with respect to both projective dimension and regularity. In particular, Bayer and Stillman [3, Corollaries 19.11 and 20.21] proved that $\operatorname{pd}(R/\operatorname{gin}(I)) = \operatorname{pd}(R/I)$ and $\operatorname{reg}(R/\operatorname{gin}(I)) = \operatorname{reg}(R/I)$. Moreover, the projective dimension of $R/\operatorname{gin}(I)$ can be read off directly from a minimal set of generators as the largest index among the indices of variables appearing in the minimal generators. (Equivalently, the projective dimension of $R/\operatorname{gin}(I)$ can be interpreted as the number of distinct variables appearing in the unique minimal generating set of the ideal).

Assume that I has minimal generators of degrees $d_1, \ldots, d_n$. The relation between the projective dimensions of R/I and $R/\operatorname{gin}(I)$ allows us to bound $\operatorname{pd}(R/I)$ in terms of $C(n, d_1 \ldots d_n)$ and the number of generators of $\operatorname{gin}(I)$ as follows:

$$\begin{aligned}
\operatorname{pd}(R/I) &= \operatorname{pd}(R/\operatorname{gin}(I)) \\
&= \text{number of variables appearing in generators of } \operatorname{gin}(I) \\
&\leq \text{sum of the degrees of generators of } \operatorname{gin}(I) \\
&\leq \text{number of generators of } \operatorname{gin}(I) \cdot \text{max generator degree of} \operatorname{gin}(I) \\
&\leq \text{number of generators of } \operatorname{gin}(I) \cdot \operatorname{reg}(R/\operatorname{gin}(I)) \\
&= \text{number of generators of } \operatorname{gin}(I) \cdot \operatorname{reg}(R/I) \\
&\leq \text{number of generators of } \operatorname{gin}(I) \cdot C(n, d_1, \ldots, d_n).
\end{aligned}$$

Hence it is sufficient to bound the number of generators of $\operatorname{gin}(I)$ in terms of $d_1, \ldots, d_n$. Since we may assume a bound on $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{gin}(I))$ is given by $C(n, d_1, \ldots, d_n)$, this means that the degrees of minimal generators of $\operatorname{gin}(I)$ are at most $C(n, d_1, \ldots, d_n)$. Note that we may assume I is already written in generic coordinates since a linear change of coordinates does not change the values of the input parameters $d_1, \ldots, d_n$.

To estimate the number of generators of the initial ideal of I, we need to understand the algorithmic procedure that produces a Gröbner basis of I, i.e. a set of elements of I whose leading terms generate the initial ideal $in(I)$. This algorithm was given by Buchberger and it involves enlarging a generating set of I by adding to the set at each step reductions of S-polynomials obtained using pairs f, g of polynomials from the preceding step's output set. The S-polynomial of a pair f, g is defined as:

$$S(f,g) = \frac{lcm(LT(f), LT(g))}{LT(f)} f - \frac{lcm(LT(f), LT(g))}{LT(g)} g.$$

Here $LT(f), LT(g)$ are the leading terms of the polynomials f, g respectively and $lcm(m_1, m_2)$ denotes the least common multiple of the monomials m_1 and m_2. Note that the degree of $S(f,g)$ is always greater or equal to the maximum of the degrees of the polynomials f, g and that equality is attained if and only if $in(f) \mid in(g)$ or $in(g) \mid in(f)$, in which case this S-polynomial need not be included in a reduced Gröbner basis. Recall that a Gröbner basis is called reduced if no monomial in any element of the basis is in the ideal generated by the leading terms of the other elements of the basis. Hence at each stage in Buchberger's algorithm the maximum degree of the polynomials obtained strictly increases. Thus the number of steps in Buchberger's algorithm is bounded by the regularity of $in(I)$, hence also by $C(n, d_1, \ldots, d_n)$.

Now starting with n minimal generators of I, in the first step in Buchberger's algorithm one computes at most $\binom{n}{2}$ S-polynomials. Similarly if one denotes the number of S-polynomials computed at the i^{th} step of Buchberger's algorithm by n_i, then $n_{i+1} \leq \binom{n_i}{2}$ and $n_1 \leq \binom{n}{2}$. Hence there is a polynomial $P(d_1, \ldots, d_N)$ such that $\sum_{i=1}^{C(d_1,\ldots,d_n)} n_i \leq P(d_1, \ldots, d_n)$. Finally one may set in this case

$$B(n, d_1, \ldots, d_n) = P(d_1, \ldots, d_n) \cdot C(n, d_1, \ldots, d_n).$$

□

It is worth noting that the bounds achieved for Questions 1.1 and 2.3 are likely quite different.

3. Upper Bounds and Special Cases

In this section we summarize the cases where the answer to Stillman's Question is known to be affirmative. In some simple cases one easily sees that a bound on projective dimension is possible. However, even with three-generated ideals in degree two, producing such a bound is nontrivial.

In the simplest case, that of $I = (f)$ being a principal ideal, $\mathrm{pd}(R/I) = 1$. If $I = (f, g)$ is minimally generated by two forms, either $\mathrm{ht}(I) = 1$ or 2. If $\mathrm{ht}(I) = 2$, then f, g is a regular sequence and $R/(f, g)$ is resolved by the Koszul complex on f and g. So $\mathrm{pd}(R/I) = 2$. If $\mathrm{ht}(I) = 1$, then there exist c, f' and g' with $f = cf'$, $g = cg'$ and $\mathrm{ht}(f', g') = 2$, so again $\mathrm{pd}(R/I) = 2$. We consider the complications for the three-generated case in the following subsection.

We also note here that when $I = (m_1, m_2, \ldots, m_n)$ is generated by n monomials, $\mathrm{pd}(R/I) \leq n$. This is clear when $n = 1$ and follows by induction by considering the short exact sequence

$$0 \longrightarrow R/((m_1, m_2, \ldots, m_{n-1}) : m_n) \xrightarrow{m_n} R/(m_1, m_2, \ldots, m_{n-1}) \longrightarrow R/I \longrightarrow 0.$$

Since $((m_1, m_2, \ldots, m_{n-1}) : m_n)$ is a monomial ideal generated by the $n-1$ monomials $\frac{\mathrm{lcm}(m_i, m_n)}{m_n}$ for $i = 1$ to $n-1$, the projective dimension of the first two terms is at most $n-1$ by induction. Hence we have $\mathrm{pd}(R/I) \leq n$, say by [18, Lemma 1]. Alternatively, one could use the fact that the Taylor resolution (see e.g. [34, Construction 26.5]) of I is a possibly non-minimal free resolution of R/I of length n. Hence the projective dimension of R/I is no larger than n.

In general, if I is generated by n polynomials each with at most m terms of degree d, then it takes at most mnd linear forms to express those n generators. Thus the projective dimension of such an ideal is at most mnd, independent of the number of variables in the ring. So all interesting cases for Stillman's Question occur when we do not assume a bound on the number of terms in each minimal generator of I.

For the rest of the section we consider the next simplest cases: three-generated ideals in low degree and ideals generated by quadratic polynomials.

3.1. **Three-Generated Ideals.** In this section we consider the projective dimension of R/I where I is minimally generated by three quadratic or three cubic forms in $K[x_1, x_2, \ldots, x_N]$. In the case of three quadratic forms, Eisenbud and Huneke proved the following in unpublished work:

Theorem 3.1 (Eisenbud-Huneke)**.** *Let $I = (f, g, h)$ where f, g and h are homogeneous minimal generators of degree 2 in a polynomial ring R. Then $\mathrm{pd}(R/I) \leq 4$.*

We will need several results to prove this theorem. Since $\operatorname{pd}(R/I)$ does not change after tensoring with an extension of the field of coefficients, we may assume that K is infinite. First we show that the multiplicity of I is at most 3. Then we handle the multiplicity 1, 2 and 3 cases separately. We begin by defining the multiplicity of an ideal and recalling related results.

For a graded module M, the *Hilbert series* $H_M(t) = \sum_{i\geq 0} dim_K M_i t^i$ can be written as a rational function of the form $H_M(t) = \frac{h(t)}{(1-t)^d}$, where d is the dimension of M and h is a polynomial of degree at most N. We define the *multiplicity* of a graded R-module M to be the value $e(M) = h(1)$. For an artinian module M, the multiplicity is equal to the *length* of the module defined as $\lambda(M) = \sum_{i\geq 0} dim_K M_i$. By convention, for a homogeneous ideal I, we refer to $e(R/I)$ as the multiplicity of I.

Next, we recall the associativity formula for multiplicity. (See e.g. [30, Theorem 14.7]). For an ideal J of R,

$$e(R/J) = \sum_{\substack{\mathfrak{p}\in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p})=\dim(R/J)}} e(R/\mathfrak{p})\lambda(R_\mathfrak{p}/J_\mathfrak{p}).$$

Let I^{un} denote the unmixed part of I, defined as the intersection of minimal primary components of I with height equal to $\operatorname{ht}(I)$. For every $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim(R/\mathfrak{p}) = \dim(R/I)$, we have that $I^{un}_\mathfrak{p} = I_\mathfrak{p}$. Hence

$$e(R/I^{un}) = \sum_{\substack{\mathfrak{p}\in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p})=\dim(R/I^{un})}} e(R/\mathfrak{p})\lambda(R_\mathfrak{p}/I^{un}_\mathfrak{p}) = \sum_{\substack{\mathfrak{p}\in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p})=\dim(R/I)}} e(R/\mathfrak{p})\lambda(R_\mathfrak{p}/I_\mathfrak{p}) = e(R/I).$$

We will often pass to the unmixed part of I and use the fact that the multiplicity does not change, as in the following result.

Proposition 3.2. *Using the notation in Theorem 3.1, if* $\operatorname{ht}(I) = 2$, *then* $e(R/I) \leq 3$.

Proof. By passing to the unmixed part of I and using structure theorems on ideals with small multiplicity, we can finish off the proof. We may assume that f, g form a regular sequence of quadratic forms. Thus $e(R/(f, g)) = 4$. We have the series of containments $(f, g) \subset I \subset I^{un}$. Note that (f, g) and I^{un} are unmixed ideals of height two. If $e(R/(f, g)) = e(R/I^{un})$, then $(f, g) = I^{un}$ by [18, Lemma 8]. But this would force $(f, g) = (f, g, h)$, contradicting that h is a minimal generator of I. Thus $4 = e(R/(f, g)) > e(R/I^{un}) = e(R/I)$. □

Following Engheta [17], we introduce the following notation to keep track of the possibilities for the associated primes of minimal height of an ideal J.

Definition 3.3. We say J is of type $\langle e = e_1, e_2, \ldots, e_m | \lambda = \lambda_1, \lambda_2, \ldots, \lambda_m\rangle$ if J has m associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of minimal height with $e(R/\mathfrak{p}_i) = e_i$ and with $\lambda(R_{\mathfrak{p}_i}/J_{\mathfrak{p}_i}) = \lambda_i$, for $i = 1, \ldots, m$. (In which case, we have $e(R/J) = \sum_{i=1}^m e_i\lambda_i$ by the associativity formula.)

We also need the following structure theorem for ideals of height two and multiplicity two.

Proposition 3.4 (Engheta [18, Prop. 11])**.** *Let J be a height two unmixed ideal of multiplicity two. Then* $\operatorname{pd}(R/J) \leq 3$ *and J is one of the following:*

(1) $(x, y) \cap (w, z) = (xw, xz, yw, yz)$ *with independent linear forms* w, x, y, z.

(2) (x, yz) *with independent linear forms* x, y, z.
(3) A prime ideal generated by a linear form and an irreducible quadratic.
(4) (x, y^2) *with independent linear forms* x, y.
(5) $(x, y)^2 + (ax + by)$ *with independent linear forms* x, y *and* $a, b \in \mathfrak{m}$ *such that* x, y, a, b *form a regular sequence.*

The proof of this proposition uses the associativity formula to divide the possibilities into one of three cases, namely: ideals of type $\langle e = 2 | \lambda = 1 \rangle$ (Case (3)), type $\langle e = 1 | \lambda = 2 \rangle$ (Cases (4) and (5)), and type $\langle e = 1, 1 | \lambda = 1, 1 \rangle$ (Cases (1) and (2)). Finally, one checks by hand that $\mathrm{pd}(R/J) \leq 3$ in each of the resulting cases.

We also need the following result, obtained originally using residual intersection techniques by Huneke and Ulrich [27], and later using more elementary homological algebra techniques by Fan [21].

Theorem 3.5 (Huneke-Ulrich [27, p.20], Fan [21, Corollary 1.2])**.** *Let* R *be a regular local ring and let* I *be a three-generated ideal of height* 2*. If* R/I^{un} *is Cohen-Macaulay (i.e.* $\mathrm{pd}(R/I^{un}) = 2$*), then* $\mathrm{pd}(R/I) \leq 3$.

For a proof, we refer the reader to [21]. This result allows us to focus only on unmixed ideals with fixed multiplicity. Using the results above, we are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Krull's (generalized) Principal Ideal Theorem [30, Theorem 13.5], $\mathrm{ht}(I) \leq 3$. If $\mathrm{ht}(I) = 1$, then there are linear forms c, f', g', h' with $f = cf'$, $g = cg'$ and $h = ch'$. Hence $I \cong (f', g', h')$, and so $\mathrm{pd}(R/I) = 3$.

If $\mathrm{ht}(I) = 3$, then f, g, h form a regular sequence and the Koszul complex on f, g, h forms a minimal free resolution of R/I. Again $\mathrm{pd}(R/I) = 3$. Hence we may assume that $\mathrm{ht}(I) = 2$. Moreover, we may assume that f, g form a regular sequence.

Now by Proposition 3.2, $e(R/I) = 1$, 2 or 3. If $e(R/I) = 1$, then by the associativity formula, I^{un} is primary to a height two prime ideal $\mathfrak{p}$ of multiplicity one. Such a prime ideal is generated by two linear forms, say $\mathfrak{p} = (x, y)$. Since $\lambda(R_\mathfrak{p}/I_\mathfrak{p}) = 1$ and I is $\mathfrak{p}$-primary, $I^{un} = \mathfrak{p}$. Clearly $\mathrm{pd}(R/\mathfrak{p}) = 2$. It then follows from Theorem 3.5 that $\mathrm{pd}(R/I) \leq 3$.

If $e(R/I) = 3$, consider the short exact sequence

$$0 \to R/((f, g) : I) \xrightarrow{h} R/(f, g) \to R/I \to 0.$$

Since f, g form a regular sequence of quadratic forms, we have $e(R/(f, g)) = 4$. Since multiplicity is additive in short exact sequences, $e(R/((f, g) : I)) = 1$. As $(f, g) : I$ is unmixed, we have $(f, g) : I = (x, y)$ for independent linear forms x and y. Therefore, $\mathrm{pd}(R/((f, g) : I)) = 2$. Since $\mathrm{pd}(R/(f, g)) = 2$, it follows that $\mathrm{pd}(R/I) \leq 3$.

Finally, in the case where $e(R/I) = 2$, we use the same exact sequence above. In this case $(f, g) : I$ is an unmixed, height two ideal of multiplicity two. By Proposition 3.4, $\mathrm{pd}(R/((f, g) : I))) \leq 3$. It follows that $\mathrm{pd}(R/I) \leq 4$. This completes the proof. □

We see from Example 2.2 that this bound is indeed tight. The next case to consider, that of an ideal minimally generated by three cubics, is significantly more complicated. In his thesis [17], and subsequently in a sequence of papers [18], [20], Engheta proved the following:

Theorem 3.6 (Engheta [20, Theorem 5])**.** *If f, g, h are three cubic forms in a polynomial ring R over a field, then* $\operatorname{pd}(R/(f, g, h)) \leq 36$.

The outline of the proof is similar to that given above for the case of three quadratic forms. Engheta first shows that the multiplicity of such an ideal is at most 8. (In characteristic zero, Engheta shows that the multiplicity can be at most 7. See [19].) He then analyzes each case separately, often using techniques from linkage theory and the structure theorems for unmixed ideals of small multiplicity. Unfortunately there is currently no complete structure theorem of unmixed ideals of multiplicity three. In those remaining cases, he shows that such ideals can be expressed in terms of a fixed number of linear or quadratic forms. A similar technique was later used by Ananyan and Hochster to study all ideals generated by linear and quadratic polynomials. For more details see Section 3.2.

We also note that the bound of 36 is likely not tight. The largest known projective dimension for an ideal minimally generated by three cubic forms is just 5. The first such example was given by Engheta. (See [20, Section 3].) The following simple example was discovered by the first author in [32].

Example 3.7. Let $R = K[a, b, c, x, y]$, where K is any field. Let $\mathfrak{m}$ denote the graded maximal ideal. Consider the ideal $I = (x^3, y^3, x^2a + xyb + y^2c)$. Then $x^2y^2 \in (I : \mathfrak{m}) - I$. It follows that $\operatorname{depth}(R/I) = 0$ and, by the Auslander-Buchsbaum theorem, that $\operatorname{pd}(R/I) = 5$.

3.2. **Ideals Generated by Quadratic Polynomials.** In a certain sense, ideals generated by quadratic polynomials are ubiquitous. In [33, Thorem 1], Mumford shows that any projective variety of degree d can be re-embedded (via the d-uple embedding) as a variety cut out by an ideal generated by quadratic forms.

In [1], Ananyan and Hochster propose a method of analyzing the projective dimension of ideals generated by polynomials of degree at most two, which need not be homogeneous. We review their idea of using a specific standard form as well as the derived recursive bound on projective dimension. Since the techniques of Ananyan and Hochster can be applied when the minimal generators are non-homogeneous, we reserve the use of the term quadratic form for a homogeneous polynomial of degree two and we call a possibly not homogeneous polynomial of degree two a quadratic polynomial. We then illustrate the techniques of [1] for the case of ideals generated by three homogeneous quadratics.

We begin with describing the focus of interest of this section: the standard form associated to an ideal generated by linear and quadratic polynomials. We note in Remark 3.10 that standard forms are by no means unique, however we shall often pick a convenient standard form and refer to it by abuse of terminology as *the* standard form associated to a certain ideal.

Definition 3.8. Let I be an ideal generated by m linear polynomials and n quadratic polynomials in a standard graded polynomial ring $R = K[x_1, \ldots, x_N]$. A standard form associated to the ideal I is given by a partition of a K-basis $\{x_1, \ldots, x_N\}$ of R_1 into subsets which satisfy the properties listed below together with a set $\{F_1, \ldots, F_{m+n}\}$ of generators of I written in a manner compatible with this partition. In the following we shall refer to the elements $\{x_1, \ldots, x_N\}$ as variables. We describe the properties required by the standard form first for the variables:

(1) The first m variables $x_1, \dots x_m$, called *leading variables*, are the linear minimal generators of I;

(2) The next $h = ht(I/(x_1, \dots x_m))$ variables $x_{m+1}, \dots x_{m+h}$, called *front variables*, are chosen such that the images $f_1, \dots, f_h$ of a maximal regular sequence $F_1, \dots, F_h$ of quadratic forms in $I/(x_1, \dots, x_m)$ under the projection $\pi : R \longrightarrow K[x_{m+1}, \dots, x_{m+h}]$ continue to form a regular sequence;

(3) The next r variables $x_{m+h+1}, \dots x_{m+h+r}$, called *primary coefficient variables*, are the coefficients of the leading and front variables when $F_1, \dots, F_n$ are viewed as polynomials in $K[x_{m+h+1}, \dots, x_N][x_1, \dots, x_{m+h}]$;

(4) The next s variables $x_{m+h+r+1}, \dots x_{m+h+r+s}$, called *secondary coefficient variables*, are the coefficients of the primary coefficient variables in the images of $F_1, \dots, F_n$ under the projection $\pi' : R \longrightarrow K[x_1, \dots, x_{m+h}]$, viewed as polynomials in $K[x_{m+h+r+1}, \dots, x_N][x_{m+h+1}, \dots, x_{m+h+r}]$;

(5) The variables $x_{m+h+r+1}, \dots, x_N$ are called the *tail variables*.

In practice, a maximal regular sequence $x_1, \dots x_m, F_1, \dots F_h$ of elements of I is chosen first and the variables $x_{m+h+1}, \dots x_N$ are obtained by extending this sequence to a system of parameters on R. From this point on we view the generators F_i as being written in terms of the variables described above. The term monomial henceforth will be used for monomials in the variables $x_1, \dots, x_N$. Next we list the properties required by the standard form for the polynomials F_i:

(1) $F_{n+i} = x_i$ for $1 \leq i \leq m$ are the linear generators of I
(2) $F_1, \dots, F_h$ form a maximal regular sequence in I
(3) $F_1, \dots, F_n$ contain no terms written using leading variables only
(4) some of the F_i may be 0 and we require that these appear last.

Partitioning the set of variables of the ring R into the various categories appearing above yields natural partitions of the monomials appearing in the generators F_i. Recall that the projection map onto the smaller polynomial ring generated by the front variables takes F_i to f_i. We call the f_i *front polynomials*. Similarly, define g_i to be the image of F_i via projecting onto the polynomial ring generated by the tail variables. We call g_i *tail polynomials*. Therefore $F_i = f_i + e_i + g_i$, where e_i is the sum of mixed terms in the leading and front, front and primary coefficient variables, leading and primary coefficient variables or primary and secondary coefficient variables and quadratic terms in the primary coefficients.

The following estimates are a clear consequence of Definition 3.8 and will prove crucial towards establishing the bound in Theorem 3.16.

Proposition 3.9 (Size estimates for the types of variables)**.** *Given I an ideal generated by m linear polynomials and n quadratic polynomials, the number of variables needed to write I in a standard form is bounded by the sum of the following estimates*

(1) exactly m leading variables, $x_1, \dots x_m$
(2) exactly $h = ht(I/(x_1, \dots x_m))$ front variables
(3) at most $n(m+h)$ primary coefficients

(4) the total number of variables needed to write the ideal of tail polynomials $g_1, \dots, g_n$ in standard form

To understand some of the subtleties of the standard form algorithm, we exhibit two examples which fit in the framework of three-generated ideals. In particular, we wish to illustrate the following:

Remark 3.10. The invariants h, m, n in Definition 3.8 are uniquely determined by I (for n to be uniquely determined one needs to assume that the set of generators of I was minimal to begin with). However the parameter $h' = \mathrm{ht}(g_1, \dots, g_n)$ may vary among different standard forms associated to the same ideal I. We note that since $(g_1, \dots, g_n)) = I/(x_1, \dots, x_{m+h+r})$, one always has $h' \le h$.

Example 3.11 (The twisted cubic)**.** Let $I_C \subset K[x_1, x_2, x_3, x_4]$ be the ideal of maximal minors of the matrix $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$. A computation shows $I_C = (x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_2x_3 - x_1x_4)$ is a prime ideal of height 2 and multiplicity 3. When thought of as an ideal in $K[x_1, x_2, x_3, x_4]$, I cuts out a curve $C \subset \mathbb{P}^3$ known as the twisted cubic.

To find a standard form one may pick x_2, x_3 as front variables. We underline the front variables in all examples for ease of parsing the respective standard forms. An inspection of the defining equations of I reveals that with respect to this choice of front variables, x_1, x_4 become primary coefficients and there are no tail variables. Following the notation introduced in 3.8, we write:

$$\begin{aligned} F_1 &= \underbrace{\underline{x_2}^2}_{f_1} - \underbrace{\underline{x_3}x_1}_{e_1} \\ F_2 &= \underbrace{\underline{x_3}^2}_{f_2} - \underbrace{\underline{x_2}x_4}_{e_2} \\ F_3 &= \underbrace{\underline{x_2x_3}}_{f_3} - \underbrace{x_1x_4}_{e_3} \end{aligned}$$

Note that $(g_1, g_2, g_3) = (0)$ and consequently $h' = 0$.

In the following we show that, regardless of the choice of the different types of variables, $h' = 0$ for any standard form of the ideal of the twisted cubic. Assume F_1, F_2 is a maximal regular sequence inside I_C. (Necessarily, F_1, F_2 will be quadratic polynomials.) Since (F_1, F_2) generate a complete intersection of multiplicity 4, $(F_1, F_1) \subset I_C$ and I_C is a prime ideal of multiplicity 3, the primary decomposition of (F_1, F_2) must be $(F_1, F_2) = I_C \cap I_L$, where I_L is a prime ideal of multiplicity 1 and height 2, i.e. the defining ideal of a line in $\mathbb{P}^3$. Let $I_L = (\ell_1, \ell_2)$ (for the choice of F_1, F_2 listed in the example above, $I_L = (x_2, x_3)$).

To extend F_1, F_2 to a maximal regular sequence on R one must pick variables $y_3, y_4 \notin I_L$. The set ℓ_1, ℓ_2, y_3, y_4 is a basis for R_1 and we shall for the moment think of the equations of I_C written in terms of this basis. Since $(F_1, F_2) \subset I_L$, F_1, F_2 are linear combinations of terms divisible either by ℓ_1 or by ℓ_2, and since (F_1, F_2) is not contained in I_L^2, some of these terms must be of the form $\ell_i y_j$ $(1 \le i \le 2, 3 \le j \le 4)$. In fact, since (F_1, F_2) is not a cone (it is in fact the union of the twisted cubic curve C and the line L), it cannot be written in terms of 3 variables only, hence both x_3 and x_4 must appear in the cross terms. We now consider any choice of front variables which will be of the form

$$y_1 = a_1\ell_1 + a_2\ell_2 + a_3y_3 + a_4y_4,\ y_2 = b_1\ell_1 + b_2\ell_2 + b_3y_3 + b_4y_4,$$

with $a_i, b_i \in K$ and $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ of rank 2. The cross terms described above yield terms of the form y_iy_j $(1 \leq i \leq 2, 3 \leq j \leq 4)$ which witness the fact that y_3, y_4 are primary coefficients. Therefore there are no tail variables and no tail polynomials in the standard form.

The reader may wonder what ideals exhibit standard forms with other values of h'. We prove a statement regarding the case $h' = 2$.

Lemma 3.12. *The only homogeneous height 2 ideal of $R = K[x_1, x_2, x_3, x_4]$ which admits a standard form with $h' = 2$ is $I = (x_1^2 + x_3^2, x_2^2 + x_4^2, x_1x_2 + x_3x_4)$.*

Proof. If I has linear generators, then the height of I modulo the leading variables is $h < \mathrm{ht}(I) = 2$. Since $h' \leq h < 2$, this contradicts the assumption $h' = 2$. Therefore there are no leading variables. Assume, up to relabeling the variables, that x_1, x_2 are the front variables. Note that the front polynomials are contained in the K-span of the monomials $\{x_1^2, x_1x_2, x_2^2\}$. Let Q be the kernel of the ring homomorphism

$$K[T_1, T_2, T_3] \xrightarrow{(f_1, f_2, f_3)} K[x_1, x_2].$$

If the vector space dimension of the K-span of f_1, f_2, f_3 is 3, then by taking suitable linear combinations of the f_i (and corresponding linear combinations of the original generators F_i) we may assume $f_1 = x_1^2, f_2 = x_2^2, f_3 = x_1x_2$. In this case $Q = (T_1T_2 - T_3^2)$. By [1, Key Lemma (c)] the tail polynomials satisfy the front relations, i.e. $g_1g_2 - g_3^2 = 0$. Since $h' = \mathrm{ht}(g_1, g_2, g_3) = 2$, the tail polynomials cannot satisfy additional relations. Thus the tail polynomials must be of the form $g_1 = l_1, g_2 = l_2, g_3 = l_1l_2$ with $l_1, l_2 \in K[x_3, x_4]_1$. But now one makes l_1, l_2 the tail variables and recovers the desired form of the ideal I.

If the vector space dimension of the K-span of f_1, f_2, f_3 is 2, then we may assume that f_1, f_2 form a regular sequence and $f_3 = 0$, hence $T_3 \in Q$. Another application of [1, Key Lemma (c)] guarantees that $g_3 = 0$. It is important to note that $h' = 2$ means at least 2 tail variables are present. The assumption that ambient ring has exactly four variables ensures there are no primary coefficients, so that $e_1 = e_2 = e_3 = 0$. But now $F_3 = f_3 + e_3 + g_3 = 0$, a contradiction. □

We are thus led to a closer examination of the ideal in the previous lemma. This example exhibits a contrasting behavior to the ideal of the twisted cubic: it allows standard forms with $h' = 0, 1$ and 2 respectively.

Example 3.13. Let $I = (x_1^2 + x_3^2, x_2^2 + x_4^2, x_1x_2 + x_3x_4) \subset \mathbb{C}[x_1, x_2, x_3, x_4]$. For this example we shall study what possible values can occur for h'.

(1) Our first choice will be to take x_1, x_2 as front variables. This makes x_3, x_4 tail variables and there are no primary (or secondary) coefficients. With the notation of Definition 3.8 we have:

$$\begin{aligned} F_1 &= \underbrace{\underline{x_1}^2}_{f_1} - \underbrace{x_3^2}_{g_1} \\ F_2 &= \underbrace{\underline{x_2}^2}_{f_2} - \underbrace{x_4^2}_{g_2} \\ F_3 &= \underbrace{\underline{x_1 x_2}}_{f_3} - \underbrace{x_3 x_4}_{g_3} \end{aligned}$$

In this case $(g_1, g_2, g_3) = (x_3^2, x_4^2, x_3x_4)$ is an ideal of height $h' = 2$.

(2) Our second choice will be to take $x_1' = x_1 + ix_3$ and x_2 as front variables. Rewriting the ideal I by substituting $x_1 = x_1' - ix_3$ yields

$$\begin{aligned} I &= ((\underline{x_1'} - ix_3)^2 + x_3^2, \underline{x_2}^2 + x_4^2, (\underline{x_1'} - ix_3)\underline{x_2} + x_3x_4) \\ &= (\underline{x_1'}^2 - 2i \cdot \underline{x_1}x_3,\ \underline{x_2}^2 + x_4^2,\ \underline{x_1'x_2} - i \cdot \underline{x_2}x_3 + x_3x_4) \end{aligned}$$

With the notation of Definition 3.8 we have:

$$\begin{aligned} F_1 &= \underbrace{\underline{x_1'}^2}_{f_1} - \underbrace{2i \cdot \underline{x_1'}x_3}_{e_1} \\ F_2 &= \underbrace{\underline{x_2}^2}_{f_2} + \underbrace{x_4^2}_{g_2} \\ F_3 &= \underbrace{\underline{x_1'x_2}}_{f_3} - \underbrace{i \cdot \underline{x_2}x_3 + x_3x_4}_{e_3} \end{aligned}$$

In this case $(g_1, g_2, g_3) = (x_4^2)$ which is an ideal of height $h' = 1$.

(3) Our last choice will be to take $y_1 = x_1 + ix_3, y_2 = x_2 + ix_4$ as front variables and we shall rename $y_3 = x_3, y_4 = x_4$. Rewriting the ideal I with respect to the linear change of coordinates from the x variables to the y variables yields:

$$\begin{aligned} I &= ((\underline{y_1} - iy_3)^2 + y_3^2, (\underline{y_2} - y_4)^2 + y_4^2, (\underline{y_1} - iy_3)(\underline{y_2} - iy_4) + y_3y_4) \\ &= (\underline{y_1}^2 - 2i \cdot \underline{y_1}y_3,\ \underline{y_2}^2 - 2i \cdot \underline{y_2}y_4,\ \underline{y_1y_2} - i \cdot \underline{y_1}y_4 - i \cdot \underline{y_2}y_3) \end{aligned}$$

With the notation of Definition 3.8 we have:

$$\begin{aligned} F_1 &= \underbrace{\underline{y_1}^2}_{f_1} - \underbrace{2i \cdot \underline{y_1}y_3}_{e_1} \\ F_2 &= \underbrace{\underline{y_2}^2}_{f_2} - \underbrace{2i \cdot \underline{y_2}y_4}_{e_2} \\ F_3 &= \underbrace{\underline{y_1y_2}}_{f_3} - \underbrace{i \cdot \underline{y_1}y_4 + i \cdot \underline{y_2}y_3}_{e_3} \end{aligned}$$

In this case $(g_1, g_2, g_3) = 0$ and consequently $h' = 0$. This behavior of the standard form can be expected since the primary decomposition

$$I = (x_1 + ix_3, x_2 - ix_4) \cap (x_1 - ix_3, x_2 + ix_4) \cap (x_1, x_2, x_3, x_4)$$

shows $I \subset (y_1, y_2)$, which means there cannot be any tail polynomials with respect to choosing y_1, y_2 as front variables.

We now sketch how the construction of standard form associated to an ideal I is applied to finding bounds for the projective dimension of R/I in [1]. The main idea is that the use of standard forms allows one to find a suitable polynomial algebra A generated by linear and quadratic forms that contains the ideal I while having a number of generators that can be bounded in terms of m, n, h. The leading, front and primary coefficient variables are all included as generators of A. Proposition 3.9 provides estimates for the respective sizes of these sets of variables. Note that it remains only to find a suitable ambient algebra for the ideal $(g_1, \dots g_n)$. The rest of the generators of A are iteratively determined, reducing the height of the ideal being analyzed.

The case when $h' < h$ allows one to use induction to complete the process. The case when $h' = h$ requires another application of the standard form for the ideal $(x_1, \dots, x_{m+h+r+s}, g_1, \dots g_n)$, which yields new front polynomials $\alpha_1, \dots, \alpha_n$ and new tail polynomials $\beta_1, \dots, \beta_n$. Let h'' be the height of the ideal $(\beta_1, \dots, \beta_n)$. If $h'' < h$ then A will be generated by the leading, front and primary coefficient variables of I together with the generators of the algebra containing $(\beta_1, \dots, \beta_n)$. If $h'' = h$ then the ideals $(f_1, \dots, f_n)$, $(g_1, \dots, g_n)$, $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n)$ are proven to be linearly presented. Towards this end one uses Lemma 3.14, the proof of which can be found in [1]. From here one deduces that there were exactly h non-zero g_i which, together with the leading, front, primary and secondary coefficient variables in the standard form of I, are the generators of A.

The following table summarizes the notations introduced. The last column refers to estimates discussed in Proposition 3.9.

Ideal	Height	Ambient ring	Number of variables
$(F_1, \dots, F_n)$	h	$S = K[x_{m+1}, \dots, x_N]$	$N - m$
$(f_1, \dots, f_n)$	h	$K[x_{m+1}, \dots, x_{m+h}]$	h
$(g_1, \dots, g_n)$	h'	$K[x_{m+h+r+s+1}, \dots, x_N]$	$N - m - h - r - s$
$(\alpha_1, \dots, \alpha_n)$	h'	$K[x_{m+h+r+s+1}, \dots, x_{m+h+r+s+h'}]$	h'
$(\beta_1, \dots, \beta_n)$	h''	$K[x_{m+h+r+s+h'+1}, \dots, x_N]$	$\leq N - m - h - r - s - h'$

Lemma 3.14 (Ananyan-Hochster [1, Lemma 4]). *Let $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ be homogeneous polynomials in two disjoint sets of indeterminates. Assume that the two sets of polynomials satisfy the same relations and denote the ideal of relations by P. Furthermore assume that the ideal of relations on the polynomials $\{\alpha_1 + \beta_1, \dots, \alpha_N + \beta_n\}$ contains P. Then P is generated by linear forms.*

We wish to give a flavor of the recursive argument by Ananyan-Hochster [1] that allows one to estimate the number of generators of A by applying their arguments to the case of ideals generated by three quadratic forms.

Proposition 3.15. *Let I be a height 2 ideal minimally generated by three quadratic forms.*

1. *If $h' = 0$, then I is an ideal in a polynomial ring generated by at most 26 linear forms.*
2. *If $h' = 1$, then I is an ideal in a polynomial ring generated by at most 30 linear forms or 26 linear forms and a quadratic form.*
3. *If $h' = 2$, then I is an ideal in a polynomial ring generated by at most 296 forms.*

Proof. (1) If $h' = 0$, then the polynomials e_i are expressible as linear combinations of x_1, x_2 with indeterminate (primary) coefficients, quadratic terms in the primary coefficients, and mixed terms in the primary and secondary coefficients. The K-vector space V of primary coefficients has dimension at most 6 (at most 2 primary coefficients appear in each of the 3 defining equations), and consequently the vector space W spanned by secondary coefficients has dimension at most $3 \cdot 6 = 18$. Since all the tail polynomials vanish, I is an ideal of the polynomial ring on variables x_1, x_2 and the union of bases of V and W.

(2) If $h' = 1$, then the previous considerations on the polynomials e_i hold and furthermore (g_1, g_2, g_3) is an ideal of height one. Therefore $(g_1, g_2, g_3) = (yy_1, yy_2, yy_3)$ for some linear forms y, y_1, y_2, y_3 (some of the y_i could be 0) or $(g_1, g_2, g_3) = (q)$ where q is an irreducible quadratic form. In the first situation I can be written in terms of x_1, x_2, at most 6 primary coefficients, at most 18 secondary coefficients, and at most 4 linear forms y, y_1, y_2, y_3, in the second case, I can be written in terms of x_1, x_2, at most 6 primary coefficients, at most 18 secondary coefficients, and one quadratic form q.

(3) If $h' = 2$, then one proceeds by putting g_1, g_2, g_3 in standard form with respect to a set of at most 18 leading variables consisting of the secondary coefficients in the standard form of the ideal I. This produces two new front variables, at most $3 \cdot 20 = 60$ new primary coefficients, and $60 \cdot 3 = 180$ secondary coefficients, a new set of front polynomials $\alpha_1, \alpha_2, \alpha_3$, and a new set of tail polynomials $\beta_1, \beta_2, \beta_3$. Let $h'' = \operatorname{ht}(\beta_1, \beta_2, \beta_3)$.

(3a) In case $h'' \leq 1$, by cases (1) and (2), the polynomials $\beta_1, \beta_2, \beta_3$ can be written in terms of at most 30 algebraically independent forms. Together with x_1, x_2, the first 6 primary coefficients, the 18 new leading variables, the 60 new primary coefficients, and the 180 new secondary coefficients, one counts 296 algebraically independent forms.

(3b) In case $h'' = 2$, Lemma 3.14 yields that there are exactly 2 non-zero g_i. We count the quantities needed to write the generators of I as follows: x_1, x_2, the first 6 primary coefficients, the 18 secondary coefficients and the two non-zero g_i. That amounts to at most 28 algebraically independent forms.

□

Applying induction on the height of the ideal in a similar manner to the proof of the Proposition above, Ananyan and Hochster obtain:

Theorem 3.16. *Let I be an ideal generated by m linear and n quadratic polynomials with $\operatorname{ht} I = h$. Then there exists a function $B(m, n, h)$ recursively defined by*

$$B(m, n, h) = (m + h)(n^3 + n^2 + n + 1) + h(n + 1) + B((m + h)n^2, n, h - 1)$$

such that I can be viewed as an ideal in a polynomial ring of at most $B(m, n, h)$ variables.

Based on this theorem and a asymptotic analysis carried out in [1] on the growth of the function $B(m, n, h)$, we conclude:

Corollary 3.17. *Stillman's question 1.1 has a positive answer in the case of ideals generated by linear and quadratic polynomials. In this case, there exists a bound on projective dimension with asymptotic order of magnitude $2(m + n)^{2(m+n)}$, where m and n are the number of linear and quadratic generators of the ideal respectively.*

We review the specific values of $B(0,3,2)$ found in Proposition 3.15.

Case	h	h'	h''	linear forms	quadratic forms	total forms
(1)	2	0	0	≤ 26	0	≤ 26
(2)	2	1	0	≤ 30	≤ 1	≤ 30
(3a)	2	2	1	≤ 296	≤ 1	≤ 296
(3b)	2	2	2	≤ 26	≤ 2	≤ 28

The bounds that are found using Theorem 3.16 are not tight. For example, compare the estimates in the previous table with the exact bound of 4 for the projective dimension of ideals generated by three quadratic forms found in Proposition 2.2.

To illustrate how the idea of counting algebraically independent variables can be improved by deeper knowledge of certain parameters associated to the ideal I, we give better bounds on the projective dimension of ideals generated by three quadratic forms using knowledge of the structure of associated primes of ideals of low multiplicity. The reader is encouraged to contrast the previous bounds to the following table in which columns 2 to 4 refer to the number of algebraically independent parameters needed to write I. The first row of the table comes from the easy observation that if $I \subset (x,y)$ with x, y linear forms, then $I = (xl_{1,1} + yl_{1,2}, xl_{2,1} + yl_{2,2}, xl_{3,1} + yl_{3,2})$ with $l_{i,j}$ linear forms (possibly 0). The second row stems from the observation that if $I \subset (x,q)$ with x a linear form and q a quadratic form, then $I = (xl_1 + aq, xl_2 + bq, xl_3 + cq)$ with l_i linear forms (possibly 0) and $a, b, c \in K$. Finally if $I \subset I_C$, the defining ideal of the twisted cubic, and I is minimally generated by three quadric forms, then $I = I_C$. The three rows exhaust the possible types of minimal associated primes of height two ideals of multiplicity at most 3 (see Proposition 3.4 for multiplicity 2 and [16] for multiplicity 3).

$Ass(I)$	linear forms	quadratic forms	total forms	$\text{pd}(R/I)$
$(x,y) \in Ass(I)$	≤ 8	0	≤ 8	≤ 8
$(x,q) \in Ass(I)$	≤ 4	1	≤ 5	≤ 5
$I = I_C$	4	0	4	2

This provides heuristic evidence that the bound in Theorem 3.16 is far from being tight. We ask in Section 6 if there is a polynomial bound on the projective dimension of ideals generated by quadratic polynomials.

4. Lower Bounds and Examples

In most cases, excepting the special cases from the previous section, there is little indication of whether the answer to Stillman's Question is affirmative or what the resulting bound would look like. One way to gain intuition into the question is to look for families of ideals with large projective dimension relative to the degrees of the generators. We present several such families in this section. Note that they neither prove nor disprove Stillman's Question, but they do provide large lower bounds on any possible answer.

An early motivation for studying Question 1.1 comes from the study of three-generated ideals. Burch [7] proved the following theorem in the local case, which was extended by Kohn [29] to the global case. We state the polynomial ring case here.

Theorem 4.1 (Burch [7], Kohn [29, Theorem A]). *Let $N \in \mathbb{N}$. There exists a polynomial ring $R = K[x_1, \dots, x_{2N}]$ and an ideal $I = (f, g, h)$ with three generators with* $\mathrm{pd}(R/I) = N + 2$.

Hence we cannot hope to find a bound on $\mathrm{pd}(R/I)$ purely in terms of the number of generators. However, if one applies this construction to a polynomial ring, the degrees of the generators grow linearly with respect to the projective dimension. Engheta computed the degrees of the three generators in the Burch-Kohn construction must be at least N, N and $2N - 2$, respectively. (See [17, Section 1.2.2].) Here is one such choice of generators.

Example 4.2. Let $R = K[x_1, \dots, x_N, y_1, \dots, y_N]$ and let

$$f = \prod_{i=1}^{N} x_i, \quad g = \prod_{i=1}^{N} y_i, \quad h = \sum_{i=1}^{N} \prod_{\substack{j=1 \\ j \neq i}}^{N} x_j y_j.$$

Then $I = (f, g, h)$ satisfies $\mathrm{pd}(R/I) = N + 2$. For example, when $N = 3$, we get the ideal

$$I = (x_1x_2x_3, y_1y_2y_3, x_2y_2x_3y_3 + x_1y_1x_3y_3 + x_1y_1x_2y_2),$$

in which case $\mathrm{pd}(R/I) = 5$ and R/I has resolution

$$R \leftarrow R^3 \leftarrow R^9 \leftarrow R^{12} \leftarrow R^6 \leftarrow R \leftarrow 0.$$

A stronger result was later proved by Bruns. He shows [6, Satz 3] that any projective resolution is the projective resolution of some three-generated ideal after modifying the first three modules in the resolution. In practice, if one constructs a three generated ideal with the same projective dimension as one with more generators, the degrees of the generators grow. If one could bound the growth of the generators when finding the 'Brunsification' of an ideal, one could reduce the study of Stillman's Question to that of three-generated ideals.

4.1. **Ideals with Large Projective Dimension.** The following example was given by the first author in [32]. A similar construction was given by Whieldon in [37].

Fix integers $m \geq 1, n \geq 0, d \geq 2$. Let $p = \frac{(m+d-2)!}{(m-1)!(d-1)!}$. Let $Z_1, \dots, Z_p$ denote all the degree $d - 1$ monomials in the variables $x_1, \dots, x_m$, ordered arbitrarily. Set $R = K[x_1, \dots, x_m, y_{1,1}, \dots, y_{p,n}]$, a polynomial ring with $m + pn$ variables over any field K. Finally, define the ideal $I_{m,n,d}$ as

$$I_{m,n,d} = \left(x_1^d, x_2^d, \dots, x_m^d, \sum_{i=1}^{p} Z_i y_{i,1}, \sum_{i=1}^{p} Z_i y_{i,2}, \dots, \sum_{i=1}^{p} Z_i y_{i,n} \right).$$

Note that $I_{m,n,d}$ has $m + n$ homogeneous generators all of degree d. The following result gives a formula for the projective dimension in terms of m, n and d.

Theorem 4.3 (McCullough [32, Theorem 3.3]). *With the notation above,*

$$\mathrm{pd}(R/I_{m,n,d}) = m + np = m + n\frac{(m + d - 2)!}{(m - 1)!(d - 1)!}.$$

The proof uses the graded Auslander-Buchsbaum theorem. (See e.g. [14, Theorem 19.9].) One shows that $\mathrm{depth}(R/I_{m,n,d}) = 0$, and hence $\mathrm{pd}(R/I_{m,n,d})$ is as large as possible.

For certain choices of m, n, d, this construction yields ideals with very large projective dimension. However, the three-generated case, where $m = 2$ and $n = 1$, yields only linear growth of projective dimension.

Example 4.4. Let $d \in \mathbb{N}$, let $R = K[x_1, x_2, y_1, y_2, \dots, y_d]$ and consider the ideals

$$I_{2,1,d} = \left(x_1^d, x_2^d, x_1^{d-1}y_1 + x_1^{d-2}x_2y_2 + \cdots + x_2^{d-1}y_d\right).$$

By the above theorem, $\operatorname{pd}(R/I_{2,1,d}) = d + 2$. Note that the cases $d = 2$ and $d = 3$ are given in Examples 2.2 and 3.7, respectively.

Example 4.5. Fix $d = 2$, $m = n \geq 2$ and let $R = K[x_1, x_2, \dots, x_n, y_{1,1}, \dots, y_{n,n}]$. Now consider the ideals

$$I_{n,n,2} = \left(x_1^2, x_2^2, \dots, x_n^2, \sum_{i=1}^n x_iy_{i,1}, \sum_{i=1}^n x_iy_{i,2}, \dots, \sum_{i=1}^n x_iy_{i,n}\right).$$

Then $I_{n,n,2}$ is generated by $2n$ quadratic polynomials and satisfies $\operatorname{pd}(R/I_{n,n,2}) = n^2 + n$. To the best of our knowledge, these are the largest projective dimension examples known for ideals generated by quadratics. So we get a lower bound of $\frac{N^2+2N}{4}$ on an answer to Stillman's Question for ideals generated by N quadratic forms - much smaller than the exponential bound achieved by Ananyan and Hochster. It would be interesting to know how close either of these bounds are to being tight.

4.2. **Ideals with Larger Projective Dimension.** In this section, we construct a family of ideals with exponentially growing projective dimension relative to the degrees of the generators, even in the three-generated case. This construction can be considered as an inductive version of the family in the previous section. The family was constructed in joint work by the two authors along with Beder, Núñez-Betancourt, Snapp and Stone in [4].

Fix integers $g \geq 2$ and a tuple of integers $m_1, \dots, m_n$ with $m_n \geq 0$, $m_{n-1} \geq 1$, and $m_i \geq 2$ for $i = 1, \dots, n-2$. We set $d = 1 + \sum_{i=1}^n m_i$. Now we define a family of sets of matrices as follows. For each $k = 0, \dots, n$, define $\mathcal{A}_k$ to be the set of $g \times n$ matrices satisfying the following properties:

(1) All entries are nonnegative integers.
(2) For $i \leq k$, column i sums to m_i.
(3) For $i > k$, column i contains all zeros.
(4) For $i \leq \min\{k, n-1\}$, column i contains at least two nonzero entries.

These matrices are used in the definition of an ideal in the standard graded ring

$$R = K[x_{i,j}, y_{\mathsf{A}} \mid 1 \leq i \leq g, 1 \leq j \leq n, \mathsf{A} \in \mathcal{A}_n].$$

Let $\mathsf{X} = (x_{i,j})$ denote a $g \times n$ matrix of variables and for every matrix $\mathsf{A} \in \mathcal{A}_n$, set $\mathsf{X}^{\mathsf{A}} = \prod_{i=1}^g \prod_{j=1}^n x_{i,j}^{a_{i,j}}$. We define the ideal $I_{g,(m_1,\dots,m_n)}$ to be

$$I_{g,(m_1,\dots,m_n)} = \left(x_{1,1}^d, \dots, x_{m,1}^d, f\right),$$

$$\text{where } f = \sum_{k=1}^{n-1} \sum_{\mathsf{A} \in \mathcal{A}_{k-1}} \sum_{j=1}^{g} \mathsf{X}^{\mathsf{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} + \sum_{\mathsf{B} \in \mathcal{A}_n} \mathsf{X}^{\mathsf{B}} y_{\mathsf{B}}.$$

With this notation, we have the following formula for the projective dimension.

Theorem 4.6 ([4, Corollary 3.3]). *Using the notation above, we have*

$$\operatorname{pd}(R/I_{g,(m_1,\dots,m_n)}) = \prod_{i=1}^{n-1}\left(\frac{(m_i+g-1)!}{(g-1)!(m_i)!} - g\right)\left(\frac{(m_n+g-1)!}{(g-1)!(m_n)!}\right) + gn.$$

As a result, one can define ideals with three generators in degree d and with projective dimension larger than $\sqrt{d}^{\sqrt{d}-1}$.

Corollary 4.7 ([4, Corollary 3.5]). *Over any field K and for any positive integer p, there exists an ideal I in a polynomial ring R over K with three homogeneous generators in degree p^2 such that $\operatorname{pd}(R/I) \geq p^{p-1}$.*

Proof. This follows directly from Theorem 4.6 by taking the ideal

$$I = I_{2,(\underbrace{p+1,\dots,p+1}_{p-1 \text{ times}},0)}.$$

$\square$

Here we give an example of a three-generated ideal I with $d = 5$ and with $\operatorname{pd}(R/I) = 8$.

Example 4.8. Consider the ideal $I_{2,(3,1)}$. Since $g = 2$ and $(m_1, m_2) = (3, 1)$, this is an ideal with three degree 5 generators. We compute the sets $\mathcal{A}_k$ first.

$$\mathcal{A}_0 = \left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\},$$

$$\mathcal{A}_1 = \left\{\begin{pmatrix}2 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}\right\},$$

$$\mathcal{A}_2 = \left\{\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\right\}.$$

Then our ring is

$$R = K\left[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right)}, y_{\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)}\right],$$

and the ideal is

$$I_{2,(3,1)} = (x_{1,1}^5, x_{2,1}^5, f),$$

where

$$\begin{aligned} f &= x_{1,1}^3x_{1,2}^2 + x_{1,1}^3x_{1,2}^2 + \mathsf{X}^{\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)}y_{\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)} + \mathsf{X}^{\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)}y_{\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)} + \mathsf{X}^{\left(\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right)}y_{\left(\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right)} + \mathsf{X}^{\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)}y_{\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)} \\ &= x_{1,1}^3x_{1,2}^2 + x_{1,1}^3x_{1,2}^2 + x_{1,1}^2x_{1,2}x_{2,1}y_{\left(\begin{smallmatrix}2&1\\1&0\end{smallmatrix}\right)} + x_{1,1}x_{1,2}x_{1,2}^2y_{\left(\begin{smallmatrix}1&1\\2&0\end{smallmatrix}\right)} \\ &\quad + x_{1,1}^2x_{2,1}x_{2,2}y_{\left(\begin{smallmatrix}2&0\\1&1\end{smallmatrix}\right)} + x_{1,1}x_{2,1}^2x_{2,2}y_{\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)}. \end{aligned}$$

Then $\mathrm{pd}(R/I_{2,(3,1)}) = 8$ and the Betti table for $R/I_{2,(3,1)}$ is shown below.

	0	1	2	3	4	5	6	7	8
total:	1	3	53	184	287	248	124	34	4
0:	1	-	-	-	-	-	-	-	-
1:	-	-	-	-	-	-	-	-	-
2:	-	-	-	-	-	-	-	-	-
3:	-	-	-	-	-	-	-	-	-
4:	-	3	-	-	-	-	-	-	-
5:	-	-	-	-	-	-	-	-	-
6:	-	-	-	-	-	-	-	-	-
7:	-	-	-	-	-	-	-	-	-
8:	-	-	3	-	-	-	-	-	-
9:	-	-	3	4	-	-	-	-	-
10:	-	-	13	46	68	56	28	8	1
11:	-	-	33	132	218	192	96	26	3
12:	-	-	1	2	1	-	-	-	-

Another interesting characteristic of this family of ideals is that it subsumes two other constructions. The construction in the previous section of ideals of the form $I_{m,1,d}$ for positive integers m, d corresponds to the ideal $I_{m,(d-1)}$ from this section, up to a relabeling of the variables. In fact, the ideals $I_{g,(m_1,\dots,m_n)}$ may be thought of as an inductive version of the ideals in the previous section. This new family of ideals also subsumes a family of ideals studied by Caviglia [8]. Let $R = K[w, x, y, z]$ and let $C_d = \left(w^d, x^d, wy^{d-1} + xz^{d-1}\right)$. Caviglia showed that $\mathrm{reg}(R/C_d) = d^2 - 2$. We first note that the ideals C_d correspond to the ideals $I_{2,(1,d-2)}$, again with a relabeling of the variables. It is also noted in [4] that some of the ideals $I_{g,(m_1,m_2,\dots,m_n)}$ have regularity larger than $d^2 - 2$. It would be interesting to compute the regularity of this new family of ideals as this would give insight into the regularity version of Stillman's Question.

5. Related Bounds

While this survey is primarily concerned with Stillman's Question, we want to mention some similar results that bound projective dimension in terms of data other than the degrees of the generators. This subsection is independent of the following sections.

Let I be an ideal of $R = K[x_1, \dots, x_N]$. A *monomial support* of I is the collection of monomials that appear as terms in a set of minimal generators of I. Note that a monomial support of an ideal is not unique. Related to Stillman's Question, Huneke asked if $\mathrm{pd}(R/I)$ was bounded by the number of monomials in a monomial support of I. If I is a monomial ideal generated by m monomials, then the monomial support of I has size m and the Taylor resolution of R/I has length m. Hence $\mathrm{pd}(R/I) \leq m$. So Huneke's question has a positive answer for monomial ideals.

In [9], Caviglia and Kummini answer Huneke's question in the negative by constructing a family of binomial ideals whose projective dimension grows exponentially relative to the size of a monomial support. In particular, for each pair of integers $n \geq 2$ and $d \geq 2$, they construct an ideal supported by $2(n-1)(d-1)+n$ monomials with projective dimension n^d. Hence they show that any upper bound for the projective dimension of an ideal supported on m monomials counted with multiplicity is at least $2^{m/2}$. These ideals also provide lower bounds on possible answers to Stillman's Question, but we present stronger examples in Section 4.

Several bounds on projective dimension for edge ideals are proven by Dao, Huneke and Schweig in [12]. Most notably, they prove that the projective dimension of the edge ideal of a graph with n vertices and maximal vertex degree d is bounded above by $n\left(1-\frac{1}{2d}\right)$ [12, Corollary 5.6]. The also prove a logarithmic bound on the projective dimension of squarefree monomial ideals of height 2 satisfying Serre's condition S_k for some $k \geq 2$. (See [12, Corollary 4.10].) Several other bounds in terms of other graph parameters are given in [13].

Finally, we mention the following result of Peeva and Sturmfels. Below $R = K[x_1, \ldots, x_N]$, $\mathcal{L}$ is a sublattice of $\mathbb{Z}^n$, and $I_{\mathcal{L}}$ is the associated lattice ideal in R, that is,

$$I_{\mathcal{L}} = \left\langle \mathbf{x}^{\mathfrak{a}} - \mathbf{x}^{\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in \mathbb{N}^n \text{ and } \mathfrak{a} - \mathfrak{b} \in \mathcal{L} \right\rangle .$$

In this setting, the projective dimension of $R/I_{\mathcal{L}}$ is bounded by an expression depending only the the height of $I_{\mathcal{L}}$.

Theorem 5.1 ([36, Theorem 2.3])**.** *The projective dimension of $R/I_{\mathcal{L}}$ as an R-module is at most $2^{\mathrm{ht}(I_{\mathcal{L}})} - 1$.*

Note that this instantly gives an answer to Stillman's Question for lattice ideals since $\mathrm{ht}(I)$ is always at most the number of minimal generators of I by Krull's generalized principal ideal theorem [30, Theorem 13.5]. However, we cannot expect such a bound in terms of $\mathrm{ht}(I)$ in general. The construction by Burch-Kohn or any of the examples in Section 4 provide examples of ideals with fixed height and unbounded projective dimension.

6. Questions

We close by posing some specific open problems related to Stillman's Question.

Question 6.1. We note that the case of an ideal I generated by three quadratics has a tight upper bound of 4 on the projective dimension of R/I. Engheta's upper bound of 36 in the case of an ideal generated by three cubics is likely far from tight. In fact, one expects that 5 is the upper bound. Can one prove this? Such a reduction will likely involve strong structure theorems on unmixed ideals of height two and low multiplicity.

Question 6.2. Similarly, Ananyan's and Hochster's exponential bound on $\mathrm{pd}(R/I)$ for ideals I generated by quadratic polynomials is likely not tight. Can one find a smaller, perhaps even polynomial bound on $\mathrm{pd}(R/I)$ where I is generated by n quadratics?

Question 6.3. There are several reductions that might make Stillman's Question more tractable. Given an ideal, can one bound the degrees of the generators of the corresponding three-generated ideal produced by Bruns' Theorem? If so, one could focus exclusively on three-generated ideals.

Question 6.4. Can one bound the projective dimension of all unmixed ideals of a given height and multiplicity? The structure theorems for ideals of height two and small multiplicity indicate that this might be possible and would provide information about Stillman's Question.

Question 6.5. Finally, we note that there are several results showing that under certain hypotheses on an ideal, one can achieve very good bounds on the regularity of the ideal in terms of the degrees of the generators. (See e.g. [2], [11] and [5].) Are any corresponding bounds possible for projective dimension?

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