SEMI-REGULAR GRAPHS OF MINIMUM INDEPENDENCE NUMBER

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Abstract. There are many functions of the degree sequence of a graph which give lower bounds on the independence number of the graph. In particular, for every graph G, $\alpha(G) \geq R(d(G))$, where R is the residue of the degree sequence of G.

We consider the precision of this estimate when it is applied to semi-regular degree sequences. We show that the residue nearly always gives the best possible estimate on independence number in the sense that, when d is semi-regular and graphic, one can always construct a graph G realizing d with $R(d) \leq \alpha(G) \leq R(d) + 1$. It is actually possible to determine explicitly, for any such d, which inequality is strict. We prove this fact directly for most semi-regular sequences, giving an outline of proof for the remainder.

1. Introduction

1.1. **Background.** Turán's famous result concerning the number of edges in a graph containing no K_r ([20]) is one of the central theorems in graph theory. By taking complements, we can regard it as giving us a lower bound on the independence number of a graph G as a function of the number of edges in G. Since the number of edges in a graph is determined by the graph's degree sequence, in fact Turán's theorem gives a lower bound on independence number as a function of the degree sequence of the graph.

There are other functions of a graph's degree sequence which relate to independence number. For example, it was shown by Caro and Wei independently in [5] and [22] that $\sum_{v \in V(G)} \frac{1}{d(v)+1}$ is a lower bound on the independence number of a graph G. Another function of the degree sequence, the "residue", is computed by repeated applications of the Havel-Hakimi reduction. It too has been shown, by Favaron, Mahéo, and Saclé in [7] and Kleitman and Griggs in [10], to bound independence number from below. We show in [15] that of the three bounds mentioned here, the residue gives the best lower estimate on independence number.

In this paper we will investigate the precision of the residue bound as it pertains to semi-regular graphs. We show that in fact the residue bound is quite good in the sense that, for any given graphic semi-regular degree sequence d, there is either a graph which realizes d and which has independence number equal to the residue of d, or, if no such graph exists, then there is a realization of d whose independence number is only one greater than the residue. Moreover, for each such sequence d, we describe a construction for a graph which realizes d with independence number as small as possible.

1.2. **Preliminaries.** All graphs in this paper are assumed to be finite and simple. A graph G is said to be *homomorphic* to another graph H if there exists a map

$\phi : V(G) \to V(H)$ such that $xy \in E(G)$ implies that $\phi(x)\phi(y) \in E(H)$. We denote the smallest degree of a vertex in G by δ and the largest degree by Δ. The *independence number* of G is the size of a largest independent set in G and is denoted by $\alpha(G)$.

A *degree sequence* is a decreasing finite sequence of natural numbers. When specifying degree sequences we write, for instance, $7^3 3^7$ when we mean the sequence of length ten consisting of three 7's and seven 3's. We denote the *degree sequence of* G by $d(G)$, which is obtained by listing the degrees of the vertices of G in descending order. If $d = d(G)$, then we denote $d(\overline{G})$ by $\overline{d}$, where $\overline{G}$ is the complement of G. A degree sequence d is said to be *graphic* if it arises as the degree sequence of some graph G; we say G *realizes* or is a *realization of* d.

The *Havel-Hakimi reduction* of d is the sequence d' obtained by dropping the first term d_1 of d, reducing the next d_1 terms of d by one, and arranging terms in descending order. It is well known that a degree sequence $d = d_1 \geq d_2 \geq \ldots \geq d_n$ is graphic if and only if its Havel-Hakimi reduction is graphic ([12]).

We let $[n] = \{1, 2, 3, \ldots n\}$. A sequence $d = d_1 d_2 \ldots d_n$ is said to be *semi-regular* if there exists k such that $d_i \in \{k, k-1\}$ for each $i \in [n]$. It is easy to tell if such a sequence is graphic:

Lemma 1.1. *The sequence* $d = k^A(k-1)^B$*,* $A > 0$*, is graphic if and only if* $A + B > k$ *and* $Ak + B(k-1)$ *is even.*

Proof. Straightforward. □

The *residue* of a graphic degree sequence is defined in terms of repeated applications of the Havel-Hakimi reduction: If d is a graphic sequence of length n, let $d^{(m)}$ denote the sequence obtained after m applications of this reduction. Then $R(d) = n - m_0$ where $m_0 = \min\{m : d^{(m)} \text{ is sequence of zeroes}\}$.

Here is an example of the computation of the residue:

$$\begin{aligned} d &= 7773333333 \\ d^{(1)} &= 663322222 \\ d^{(2)} &= 52222111 \\ d^{(3)} &= 1111110 \\ d^{(4)} &= 111100 \\ d^{(5)} &= 11000 \\ d^{(6)} &= 0000 \\ \text{Thus } R(7^3 3^7) &= 4. \end{aligned}$$

This iterative process for computing $R(d)$ can be avoided if d is semi-regular. In this case, we have the following closed formula:

Theorem 1.2. *Let* $d = k^A(k-1)^B$ *be a graphic sequence. Then* $R(d) = \left\lceil \frac{A}{k+1} + \frac{B}{k} \right\rceil$.

Proof. See [7]. □

The program Graffiti was the first to conjecture that the residue of the degree sequence gives a lower bound on a graph's independence number. Favaron, Mahéo and Saclé proved this fact in [7]; Kleitmann and Griggs gave a simpler proof later in [10].

Theorem 1.3. *For any graph G, $\alpha(G) \geq R(d(G))$.*

For any graphic sequence d, we define $\alpha(d) := \min\{\alpha(G) : d(G) = d\}$. That $\alpha(d) \geq R(d)$ is immediate by Theorem 1.3. Thus if G realizes d with $\alpha(G) = R(d)$, then $\alpha(d) = R(d)$. We will call such a realization of d *optimal*.

The aim of this paper is to show that, for d graphic and semi-regular, optimal realizations of d usually exist. More precisely, $R(d) \leq \alpha(d) \leq R(d) + 1$ and the strictness of the inequalities can be determined explicitly from d.

2. Cores of Semi-Regular Sequences

Let $d = k^A(k-1)^B$ be a graphic semi-regular sequence. Then there exist integers a, b and m, n such that

$$\begin{array}{lll} A = m(k+1) + a, & 0 \leq a < k+1, & m \geq 0 \\ B = nk + b, & 0 \leq b < k, & n \geq 0. \end{array}$$

Writing d as

$$d = k^{m(k+1)+a}(k-1)^{nk+b} = k^{m(k+1)}k^a(k-1)^b(k-1)^{nk}$$

we notice that if both $a = 0$ and $b = 0$, then d has an optimal realization: the graph which consists of m disjoint copies of K_{k+1} together with n disjoint copies of K_k has independence number $m + n$, the value of $R(d)$ in this case. This observation suggests that, for a or b nonzero, we focus on finding an optimal realization of the "remainder" of the sequence, adjoining disjoint copies of complete graphs to it in order to achieve an optimal realization of the sequence overall. This idea motivates the definition below.

Henceforth, for a graphic sequence $d = k^A(k-1)^B$, a, b, m and n will always be taken as defined above. Note that $Ak + B(k-1)$ is even if and only if $ak + b(k-1)$ is even.

Definition 2.1. If $d = k^{m(k+1)+a}(k-1)^{nk+b}$ is graphic, then the *core of* d is defined to be $d_C = k^a(k-1)^b$.

If $m \geq 1$ we say that d *goes left* and define the *left core* to be $d_L = k^{k+1+a}(k-1)^b$. If d goes left and $m \geq 2$, we define $d_{LL} = k^{2(k+1)+a}(k-1)^b$.

If $n \geq 1$ we say that d *goes right* and define the *right core* to be $d_R = k^a(k-1)^{b+k}$. If d goes right and $n \geq 2$, we define $d_{RR} = k^a(k-1)^{b+2k}$.

If $d = d_L$ and d_C is not graphic we say that d is *left minimal*.

If $d = d_R$ and d_C is not graphic we say that d is *right minimal*.

Notice that the core of d need not be graphic. For example, $d = 5^8 4^{13} = 5^6 5^2 4^3 4^{10}$ has core $d_C = 5^2 4^3$, which is not graphic, while both $d_L = 5^8 4^3$ and $d_R = 5^2 4^8$ are graphic.

We state some facts about cores, noting the following corollary to Lemma 1.1:

Corollary 2.2. *Let $d = k^A(k-1)^B$ be graphic. If d_C is non-empty, then d_C is graphic provided it is long enough. If d_C is not graphic, then either d goes left or d goes right. If d goes left, d_L is graphic, and if $m \geq 2$ then d_{LL} is graphic. If d goes right, d_R is graphic, and if $n \geq 2$ then d_{RR} is graphic.*

Proof. This follows directly from Lemma 1.1 and Definition 2.1. □

Proposition 2.3. *If d_C is non-empty and graphic then $R(d_C) = 2$.*

Proof. By Theorem 1.2, $R(d_C) = \left\lceil \frac{a}{k+1} + \frac{b}{k} \right\rceil$. Note that d_C non-empty and graphic implies that neither a nor b is zero so that $a + b > k$. We have that

$$2 > \frac{2k-1}{k} \geq \frac{a+b}{k} > \frac{a}{k+1} + \frac{b}{k} > \frac{a+b}{k+1} \geq \frac{k+1}{k+1} = 1$$

Hence $R(d_C) = 2$. □

The following proposition and its corollary are obvious and are used in many of the constructions which follow. We give proof in order to introduce notation which will be used later.

Proposition 2.4. *The edges of $K_{n,n}$ can be partitioned into n disjoint perfect matchings.*

Proof. Let $V(K_{n,n}) = X \cup Y$. Label the vertices of X as $\{x_1, \ldots, x_n\}$ and the vertices of Y as $\{y_1, \ldots, y_n\}$. Let π denote the permutation $(12 \ldots n)$ and define

$$P_j[X,Y] = \{x_{\pi^j(i)} y_i : i = 1, 2, 3, ..., n\}.$$

Then, for each $j \in [n]$, $P_j[X,Y] \subset E(K_{n,n})$ is a perfect matching and, for $(i,j) \in [n] \times [n]$ such that $i \neq j$, we have that $P_i[X,Y] \cap P_j[X,Y] = \emptyset$. Clearly $\bigcup_{j=1}^{n} P_j[X,Y] = E(K_{n,n})$ so that $\{P_j[X,Y] : j = 1, 2, 3, ..., n\}$ forms a partition of $E(K_{n,n})$. □

Recall that a matching M in a graph G is said to be *semi-perfect* if exactly one vertex of G is not incident with any edge in M.

Corollary 2.5. *The edges of $K_{n+1,n}$ can be partitioned into $n+1$ semi-perfect matchings.*

Proof. Apply Proposition 2.4 to $K_{n+1,n+1}$. Removing any vertex x in this graph yields the graph $K_{n+1,n}$ with the edges partitioned into $n+1$ semi-perfect matchings as claimed. □

Proposition 2.6. *If $d = k^C(k-1)^D$ is graphic and $C + D = 2l$ for some $l \in \mathbb{N}$ such that $l \leq k$ then there exists a realization G of d with $\alpha(G) \leq 2$.*

Proof. The conditions from Lemma 1.1 imply that $0 \leq k - l < l$. Since d is graphic and C and D have the same parity, both C and D are even. Build a graph G in the following way: Start with two disjoint copies of K_l and insert $k - l$ disjoint perfect matchings, together with $\frac{C}{2}$ edges of another matching, between their vertex sets. Then $d(G) = ((l-1) + (k-l) + 1)^C ((l-1) + (k-l))^{2l-C} = k^C(k-1)^D$. Since $V(G)$ is partitioned into two parts each of which induce a clique in G, we have that $\alpha(G) \leq 2$. □

Lemma 2.7. *If d_C is nonempty and graphic then $\alpha(d_C) = R(d_C)$.*

Proof. By Proposition 2.3, it is enough to present a graph realizing d_C with independence number 2.

If the length of $d_C = k^a(k-1)^b$ is even, we are done by Proposition 2.6 and Theorem 1.3.

If the length of d_C is odd, we can write $a + b = 2l + 1$ for some integer $0 < l < k$. Lemma 1.1 and the fact that a and b have opposing parities imply that b and k have the same parity; thus we can write $b = k - 2j$ for some integer $0 < j < k - l$. Note that $k < 2l + 1 \leq 2k - 1$ so that $0 < k - l < l + 1$.

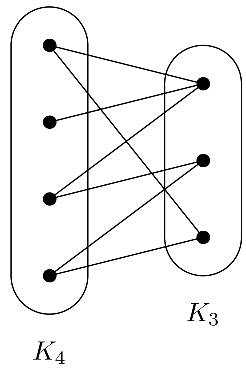


FIGURE 1. The sequence $d_C = 5^4 4^3$ is graphic and has odd length.

We build a graph G in the following way: starting with a copy of K_{l+1} and a copy of K_l, we first insert $k-l$ disjoint semi-perfect matchings between their vertex sets, then select, from another matching, j edges whose endpoints in K_{l+1} are left unmatched by one of the matchings we already included. Then $d(G) = k^{(2l+1)-(k-2j)}(k-1)^{k-2j} = k^a(k-1)^b$. Obviously $\alpha(G) \le 2$. □

An example of this construction is given in Figure 1. (Note that in the figure the edges inside the K_4 and K_3 are not shown.) If d_C is not graphic, similar constructions can be described for d_L and d_R, provided they have even length.

Proposition 2.8. *If d_C is not graphic and d goes left, then $R(d_L) = 2$. If d_C is not graphic and d goes right, then $R(d_R) = 2$.*

Proof. By Theorem 1.2, $R(d_L) = \left\lceil \frac{k+1+a}{k+1} + \frac{b}{k} \right\rceil = 1 + \left\lceil \frac{a}{k+1} + \frac{b}{k} \right\rceil$ and $R(d_R) = \left\lceil \frac{a}{k+1} + \frac{b+k}{k} \right\rceil = 1 + \left\lceil \frac{a}{k+1} + \frac{b}{k} \right\rceil$. Note that $0 < a+b \le k$, since d_C is not graphic, and thus

$$0 < \frac{a}{k+1} + \frac{b}{k} \le \frac{a+b}{k} \le \frac{k}{k} \le 1.$$

Thus $\left\lceil \frac{a}{k+1} + \frac{b}{k} \right\rceil = 1$ and so $R(d_L) = 2$ and $R(d_R) = 2$. □

Lemma 2.9. *If d_C is not graphic but d goes left and d_L has even length then $\alpha(d_L) = R(d_L)$. If d_C is not graphic but d goes right and d_R has even length then $\alpha(d_R) = R(d_R)$.*

Proof. Again, $0 < a+b \le k$, since d_C is not graphic. Thus the conditions of Proposition 2.6 are met with respect to the sequences $d_L = k^{k+1+a}(k-1)^b$ and $d_R = k^a(k-1)^{b+k}$. By Theorem 1.3 and Proposition 2.8 we are done. □

The constructions described above can be augmented with complete graphs to obtain optimal realizations of most semi-regular sequences:

Theorem 2.10. *If d is graphic then $\alpha(d) = R(d)$ provided at least one of the following holds:*

1. *d_C is graphic.*
2. *d_C is not graphic but d goes left and d_L has even length.*
3. *d_C is not graphic but d goes right and d_R has even length.*

Proof. Suppose d_C is graphic. If d_C is empty, we noted at the beginning of the section that d has an optimal realization. Assuming d_C is non-empty, we have that

$$\begin{aligned}R(d) &= \left\lceil \frac{A}{k+1} + \frac{B}{k} \right\rceil \\ &= \left\lceil \frac{m(k+1)+a}{k+1} + \frac{b+nk}{k} \right\rceil \\ &= m + \left\lceil \frac{a}{k+1} + \frac{b}{k} \right\rceil + n \\ &= m + R(d_C) + n.\end{aligned}$$

Let G be the optimal realization of d_C as in Lemma 2.7 so that $\alpha(G) = R(d_C)$. Define

$$H = \bigcup_{i=1}^{m} K_{k+1} \cup G \cup \bigcup_{i=1}^{n} K_k.$$

Then $d(H) = d$ and clearly $\alpha(H) = m + \alpha(G) + n = R(d)$. Hence $\alpha(d) = R(d)$ in this case. The other two cases are treated in a similar manner. □

Theorem 2.10 applies to most semi-regular sequences since, if d goes both left and right, then either d_L or d_R has even length. Thus we are left to consider only those semi-regular graphic sequences d such that d_C is not graphic and such that d goes left but not right and d_L has odd length, or such that d goes right but not left and d_R has odd length. We will examine these situations in more detail soon. We conclude this section by proving an upper bound on $\alpha(d)$ for any semi-regular graphic sequence d.

Lemma 2.11. *If d_C is not graphic but d goes left and d_L has odd length then $\alpha(d_L) \leq 3$.*

Proof. Recall that $d_L = k^{k+1+a}(k-1)^b$. Write the length $k+1+a+b = 2l+1$ for some $l \in \mathbb{N}$. Notice that, since d_C is not graphic, $0 < a+b \leq k$ and we have $k < 2l \leq 2k$ so that $0 \leq k-l < l$. Lemma 1.1 together with the odd length of D_L tell us that $k(k+1)+ak+b(k-1)$ is even and $k+1+a+b$ is odd. If a were odd the second fact would tell us that b was not congruent to k modulo 2, and the first fact would tell us that b was congruent to k modulo 2. This contradiction proves that a is even.

We construct a graph G in the following way: start with two disjoint copies of K_l and insert $k-l$ disjoint perfect matchings between their vertex sets, together with $\frac{a}{2}$ edges of an additional disjoint perfect matching M. Now adjoin a new vertex x to k vertices which are not incident to any of the $\frac{a}{2}$ edges of M. This can be done since $2l - a = k + b \geq k$. It is easy to see that

$$\begin{aligned}d(G) &= k^1((l-1)+(k-l)+1)^k((l-1)+(k-l)+1)^a((l-1)+(k-l))^{2l-a-k} \\ &= k^{k+1+a}(k-1)^b.\end{aligned}$$

Clearly, $\alpha(G) \leq 3$. □

An example of the above construction is illustrated in Figure 2. We have a similar result when d goes right:

Lemma 2.12. *If d_C is not graphic but d goes right and d_R has odd length then $\alpha(d_R) \leq 3$.*

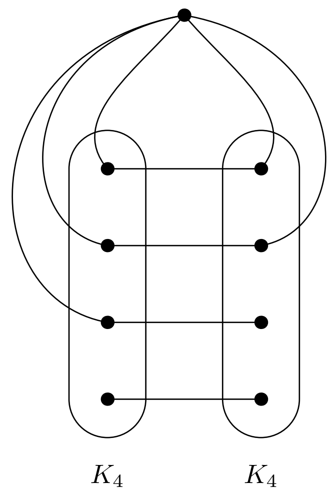


FIGURE 2. $d_L = 5^6 4^3$ has odd length and the core $d_C = 5^0 4^3$ is not graphic.

Proof. Recall that $d_R = k^a(k-1)^{b+k}$. Write the length $a+b+k = 2l+1$ for some $l \in \mathbb{N}$ and notice that, since $0 < a+b \leq k$, we have that $0 < k-l < l+1$. Also, by Lemma 1.1, we must have that k and a have opposing parities so that $a < k$ and $\frac{k-(a-1)}{2} \in \mathbb{N}$.

Assume first that $a = 0$. This implies that k is odd. Construct a graph G by taking two disjoint copies of K_l and include $k-l-1$ perfect matchings between their vertex sets, together with all but $\frac{k-1}{2}$ edges of another matching M. Join a new vertex x to those $k-1$ vertices which are not incident to any included edge of M. Then every vertex has degree $k-1$ and clearly $\alpha(G) \leq 3$.

Now if $a > 0$ build G similarly by taking two disjoint copies of K_l but now include $k-l-1$ perfect matchings between vertex sets together with all but $\frac{k-(a-1)}{2}$ edges of another matching M. Join a new vertex x to a total of k vertices, including the $k-(a-1)$ vertices missed by the chosen edges of M. Then $d(G) = k^a(k-1)^{b+k}$ and clearly $\alpha(G) \leq 3$. □

Theorem 2.13. *If d is semi-regular and graphic then $\alpha(d) \leq R(d)+1$.*

Proof. This follows from Proposition 2.8, Lemmas 2.11 and 2.12, and the technique used in the proof of Theorem 2.10. □

3. A Tripartite Construction

Together, Theorems 1.3 and 2.13 show that $R(d) \leq \alpha(d) \leq R(d)+1$ for all semi-regular graphic sequences d. By Theorem 2.10 the left inequality is in fact seen to be equality in many instances. Indeed, the only sequences $d = k^{m(k+1)+a}(k-1)^{b+nk}$ for which we have not determined which inequality is strict are those such that d_C is not graphic and either $m = 0$ and d_R has odd length, or $n = 0$ and d_L has odd length. The remainder of the paper focuses on these sequences.

In this section, we determine the strictness of Theorem 2.13 for those graphic semi-regular sequences $d = k^{m(k+1)+a}(k-1)^{b+nk}$ where d_C is not graphic but either

$m \geq 2$ or $n \geq 2$. This will leave only the left and right minimal sequences of odd length to discuss.

Our strategy is similar to that employed in the previous section: We present constructions for optimal realizations of d_{RR} and d_{LL} and adjoin copies of complete graphs to obtain an optimal realization of the sequence d as a whole. We will need the following definition and lemmas.

Definition 3.1. An ordered pair of sequences (a, b), where $a = a_1 a_2 \dots a_n$, $b = b_1 b_2 \dots b_m$, is said to be *bigraphic* if there exists a bipartite graph G with $V(G) = X \cup Y$ where a lists the degrees of the vertices in X in decreasing order and b lists the degrees of the vertices in Y in decreasing order.

The *reduction of* (a, b) is denoted by (a', b') where a' is the sequence of length $n-1$ obtained by deleting the largest entry Δ from a and b' is the sequence of length m obtained by reducing the Δ largest entries of b by one.

It is an exercise to show that (a, b) is bigraphic if and only if (a', b') is. If a and b are semi-regular, we have another test for determining whether or not (a, b) is bigraphic:

Lemma 3.2. *Let $a = (a_i)_{i=1}^n$ and $b = (b_j)_{j=1}^m$ be semi-regular sequences of nonnegative integers such that*

$$a_i \leq m \;\forall i \in [n]$$
$$b_j \leq n \;\forall j \in [m]$$
$$\sum_{i=1}^n a_i = \sum_{j=1}^m b_j.$$

Then (a, b) is bigraphic.

Proof. We induct on $n+m$, noting that the case where $n+m = 0$ is trivial. Assume $n+m > 0$ and that, without loss of generality, $n \geq m$, $a_1 \geq a_2 \geq \dots \geq a_n$, and $b_1 \geq b_2 \geq \dots \geq b_m$.

If $a_1 = 0$ then $\sum_{i=1}^n a_i = 0$ so that the graph on $n+m$ isolated vertices shows (a, b) is bigraphic.

If $a_1 > 0$, notice that $|\{j : b_j > 0\}| \geq a_1$. Recall that a' is the sequence of length $n-1$ defined as

$$a'_i = a_{i+1} \;\forall i \in [n-1]$$

and b' is the sequence obtained by reducing the first a_1 terms of b by one:

$$b'_j = b_j - 1 \;\forall j \leq a_1$$
$$b'_j = b_j \;\forall j > a_1.$$

Clearly, a' and b' are semi-regular. Also,

$$\sum_{i=1}^{n-1} a'_i = \left(\sum_{i=1}^n a_i\right) - a_1 = \left(\sum_{j=1}^m b_j\right) - a_1 = \sum_{j=1}^m b'_j$$

and

$$a'_i = a_{i+1} \leq m \;\forall i \in [n-1].$$

In order to show that a' and b' satisfy the hypotheses of the theorem, we need only verify that, for all $j \in [m]$, we have that $b'_j \leq n-1$. If there exists $k \in [m]$ such that $b'_k > n-1$ then we must have that $b'_k = n$ and that $b'_k = b_k$, since $b'_j \leq b_j \leq n$. Thus $b_j \in \{n, n-1\}$ for each j and $a_1 < k \leq m$. But then

$$\sum_{j=1}^{m} b_j > m(n-1) + a_1 > mn - m \geq mn - n = n(m-1) \geq \sum_{i=1}^{n} a_i,$$

contrary to assumption. Thus by induction (a', b') is bigraphic and so there exists a bipartite graph $G' = G'[X', Y']$ where a' lists the degrees of vertices in X' while b' lists the degrees of vertices in Y'. We obtain a new graph $G = G[X, Y]$ where

$$X = X' \cup \{x\}$$
$$Y = Y'$$

by adjoining a new vertex x to those a_1 vertices of Y' whose degrees correspond to the numbers b'_j where $j \in [a_1]$ so that (a, b) is seen to be bigraphic. □

Lemma 3.3. *Given vertex weights a, b, c on the vertices of K_3, there exist edge weights a', b', c' (where a' is the weight on the edge opposite the vertex of weight a, etc.) such that the weight at a vertex is the sum of the weights of the incident edges.*

Proof. The weights are:

$$a' = \frac{1}{2}(-a + b + c)$$
$$b' = \frac{1}{2}(a - b + c)$$
$$c' = \frac{1}{2}(a + b - c).$$

□

Lemma 3.4. *Suppose $d = d_1 d_2 d_3 \ldots d_n$ is a semi-regular degree sequence such that*

$$\sum_{i=1}^{n} d_i = D.$$

For $D_1, D_2 \in \mathbb{N}$ such that $D_1 + D_2 = D$ there exist semi-regular sequences (not necessarily in decreasing order) d' and d'' of length n such that

$$d_i = d'_i + d''_i$$

for each i and such that

$$\sum_{i=1}^{n} d'_i = D_1 \text{ and } \sum_{i=1}^{n} d''_i = D_2.$$

Proof. Straightforward. □

We are now poised to prove the following theorem.

Theorem 3.5. *Suppose $d = (d_i)_{i=1}^k, e = (e_i)_{i=1}^l, f = (f_i)_{i=1}^m$ are all semi-regular degree sequences of nonnegative integers and that*

$$D = \textstyle\sum_{i=1}^k d_i, \qquad E = \sum_{i=1}^l e_i, \qquad F = \sum_{i=1}^m f_i.$$

Then there exists a tripartite graph G with parts $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ such that $|\mathcal{D}| = k$, $|\mathcal{E}| = l$, and $|\mathcal{F}| = m$, where d lists the degrees of vertices of G in $\mathcal{D}$, e lists the *degrees of the vertices in $\mathcal{E}$, and f lists the degrees of the vertices in $\mathcal{F}$ if and only if the following hold:*

1. $D + E + F$ *is even*
2. $D \le E + F$
   $E \le D + F$
   $F \le E + D$
3. $0 \le D' \le lm$
   $0 \le E' \le km$
   $0 \le F' \le kl$

where D', E', F' are as in Lemma 3.3 with respect to the weights D, E, and F. In such a realization D' is the number of edges between $\mathcal{E}$ and $\mathcal{F}$, E' is the number of edges between $\mathcal{D}$ and $\mathcal{F}$, and F' is the number of edges between $\mathcal{D}$ and $\mathcal{E}$.

Proof. The three conditions are necessary, for if G is such a graph then $\frac{D+E+F}{2}$ counts the edges of G, the number of edges leaving one part can not exceed the total number of edges leaving the other two parts, and the number of edges between any two parts is no more than the product of the sizes of those parts.

Now we prove sufficiency. Since $E' + F' = D$, by Lemma 3.4 there exist semi-regular sequences d^E and d^F (not necessarily in decreasing order) such that $d_i^E + d_i^F = d_i$ for each $i \in [k]$ and

$$\sum_{i=1}^{k} d_i^E = F', \qquad \sum_{i=1}^{k} d_i^F = E'.$$

Similarly, since $D' + F' = E$ and $E' + D' = F$, there exist semi-regular sequences e^D and e^F such that $e_i^D + e_i^F = e_i$ for each $i \in [l]$ with

$$\sum_{i=1}^{l} e_i^D = F', \qquad \sum_{i=1}^{l} e_i^F = D'$$

and semi-regular sequences f^D and f^E such that $f_i^D + f_i^E = f_i$ for each $i \in [m]$ with

$$\sum_{i=1}^{m} f_i^D = E', \qquad \sum_{i=1}^{m} f_i^E = D'.$$

We focus for the moment on the sequences d^E and e^D. Since $\sum_{i=1}^{l} e_i^D = F' = \sum_{i=1}^{k} d_i^E$ we will apply Lemma 3.2 to the sequences e^D and d^E. We need only check that $d_i^E \le l$ for each $i \in [k]$ and that $e_i^D \le k$ for each $i \in [l]$. But this is easy since $F' \le kl$, by hypothesis. Thus, the average value of an entry in d^E is at most $\frac{kl}{k} = l$. Since d^E is semi-regular we must have that $d_i^E \le l$. Similarly the average value of an entry in e^D is at most $\frac{lk}{l} = k$ and e^D is semi-regular so that $e_i^D \le k$. Thus Lemma 3.2 does apply and so there exists a bipartite graph $G_1 = G[X, Y]$ with $|X| = l$, $|Y| = k$ such that e^D lists the degrees of vertices in X and d^E lists the degrees for vertices in Y.

Similar arguments hold with respect to the pairs of sequences d^F and f^D, and e^F and f^E. We obtain a bipartite graph $G_2 = G[Y', Z]$ with $|Y'| = k$, $|Z| = m$ such that d^F lists the degrees of vertices in Y' and f^D lists the degrees of vertices in Z, and a third bipartite graph $G_3 = G[Z', X']$ such that $|Z'| = m$, $|X'| = l$ and f^E lists the degrees of vertices in Z' while e^F lists the degrees of vertices in X'.

We create a tripartite graph T by gluing the three bipartite graphs G_1, G_2, and G_3 together by identifying the vertex class X with X', the class Y with Y', and the class Z with Z' in the obvious way so that the degrees of the vertices in X are listed by the sequence $e^D + e^F = e$, the degrees in Y are listed by the sequence $d^E + d^F = d$, and the degrees of the vertices in Z are listed by the sequence $f^E + f^D = f$. □

The next two lemmas show that the tripartite graph T of Theorem 3.5 can be extended to an optimal realization of d_{LL} and d_{RR} for all but two values of $a + b$.

Lemma 3.6. *Suppose that $d = k^{m(k+1)+a}(k-1)^{b+nk}$ is graphic with $0 < a+b < k$ and that d goes left with $m \geq 2$. Then $\alpha(d_{LL}) = R(d_{LL})$.*

Proof. Recall that $d_{LL} = k^{2(k+1)+a}(k-1)^b$ and observe that the condition on $a + b$ implies that $R(d_{LL}) = 3$ and $k \geq 2$. We will write the length of d_{LL} as $2(k+1) + a + b = 3l + i$ for some $l \in \mathbb{N}, 0 \leq i < 3$. Note that $l \geq 2$.

In the table that follows we list, for each value of i, semi-regular degree sequences d, e, and f together with the values D, E, F, D', E' and F' that they determine as in Theorem 3.5 and Lemma 3.3, as well as any necessary caveats. By applying Theorem 3.5, we obtain a tripartite graph G_i which we augment to obtain an optimal realization G of d_{LL} (i.e. $\alpha(G) = 3$).

	$i=0$	$i=1$	$i=2$
d	$(k-l+1)^{l-b}(k-l)^b$	$(k-l)^{l+1-b}(k-l-1)^b$	$(k-l+1)^{l-b}(k-l)^b$
$e=f$	$(k-l+1)^l$	$(k-l+1)^l$	$(k-l)^{l+1}$
D	$l(k-l)+(l-b)$	$(l+1)(k-l)-b$	$l(k-l)+(l-b)$
$E=F$	$l(k-l)+l$	$l(k-l)+l$	$(l+1)(k-l)$
$F'=E'$	$\frac{1}{2}D$	$\frac{1}{2}D$	$\frac{1}{2}D$
$D'=E-\frac{1}{2}D$	$\frac{1}{2}[l(k-l)+l+b]$	$\frac{1}{2}[(l-1)(k-l)+2l+b]$	$\frac{1}{2}[(l+2)(k-l)-l+b]$
caveats		$k>l$	$k>l$

It is straightforward to check that the following relationships hold:

$$\frac{3}{2}l > k \geq l > b.$$

These relationships, together with the provision $k > l$ when $i = 1$ as given in the table, are enough to show that $D \geq 0$ throughout.

The caveats of the table are actually benign. In the case $i = 1$, if $k = l$ we have that d_{LL} has length $2k + 2 + a + b = 3k + 1$. This implies $a + b = k - 1$ so that the length of d_L is even. By Theorem 2.10, $\alpha(d_{LL}) = R(d_{LL})$ in this case. In the case $i = 2$, if $k = l$ we have that the length of d_{LL} satisfies $2k + 2 + a + b = 3k + 2$ so that $a + b = k$, but this is ruled out by hypothesis. We will therefore assume the caveats hold.

We need to verify that $D + E + F$ is even. Notice that, for each i, $D + E + F = D + 2E \equiv D \pmod 2$ so that we need only check that D is even. Observe that $2(k+1)+a+b = 3l+i$ so that $a+b \equiv l+i \pmod 2$. Thus $a+b$ has the same parity as l when $i = 0, 2$ and the opposite parity as l when $i = 1$. Using this observation and the fact that d_{LL} is graphic, we summarize, (mod 2), all the possible parity combinations in the table below.

$i = 0, 2$

k	a	b	l
0	0	0	0
0	1	0	1
1	0	0	0
1	0	1	1

$i = 1$

k	a	b	l
0	0	0	1
0	1	0	0
1	0	0	1
1	0	1	0

In every case we see that D is even so that the first condition of Theorem 3.5 is satisfied.

To check the lower bounds of the third condition, recall by Lemma 3.3 that $F' = \frac{1}{2}(D+E-F)$ and $E' = \frac{1}{2}(D-E+F)$. Thus $F' = E' = \frac{1}{2}D \geq 0$. It is easy to check that $D' = \frac{1}{2}(-D + E + F) = E - \frac{1}{2}D \geq 0$ with our stated assumptions. We check also that the upper bounds on D',E' and F' all hold: For $i = 0$ we require $E' = F' \leq l^2$ and $D' \leq l^2$; for $i = 1$ we require $E' = F' \leq l(l+1)$ and $D' \leq l^2$; and for $i = 2$ we require that $E' = F' \leq l(l+1)$ and $D' \leq (l+1)(l+1)$. Notice all of these inequalities are true for $l \geq 2$.

Finally, since $D \geq 0$, we do have $E \leq E + D = F + D$ and $F \leq F + D = E + D$. Since we've seen that $D' = E - \frac{1}{2}D \geq 0$ we have that $D \leq 2E = E + F$ so that the second condition of Theorem 3.5 holds.

Thus, the conditions of the theorem are met for each value of i. In each case we are guaranteed a tripartite graph T_i with parts $X, Y,$ and Z such that $d, e,$ and f list the degrees of vertices in X, Y and Z respectively. For a fixed value of i, obtain a new graph G by adding to T_i all edges joining two vertices within X, Y, or Z so that each of these sets induces a clique in G. Then $\alpha(G) \leq 3$ and we have that

$$d(G) = k^{3l+i-b}(k-1)^b = k^{2(k+1)+a}(k-1)^b.$$

Now $\alpha(G) \geq R(d_{LL}) = 3$ ensures that $\alpha(G) = 3$. Thus, the lemma holds. □

Figure 3 depicts the graph obtained by this construction for the sequence $d = 4^{10}3^2$.

Lemma 3.7. *Suppose that* $d = k^{m(k+1)+a}(k-1)^{b+nk}$, $a < k+1$ *and* $b < k$, *is graphic with* $0 < a + b < k - 1$ *and that* d *goes right with* $n \geq 2$. *Then* $\alpha(d_{RR}) = R(d_{RR})$.

Proof. The method of proof is the same as for the previous lemma. As before we present, for each value of i, semi-regular degree sequences d, e, and f and the values

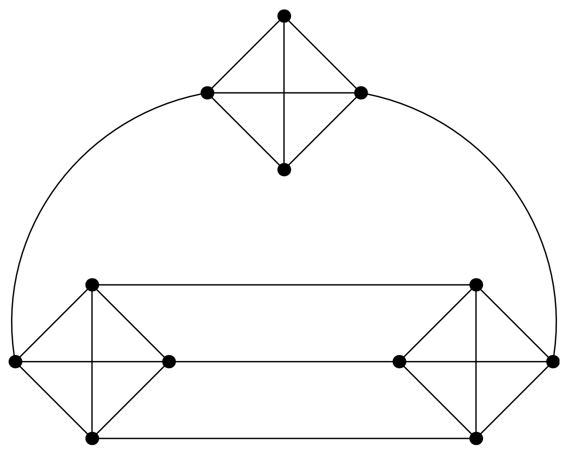


FIGURE 3. The construction of Lemma 3.6 where $d_{LL} = 4^{10}3^2$.

D, E, F, D', E' and F' they determine, as in Lemma 3.3 and Theorem 3.5. One checks that Theorem 3.5 applies, thereby obtaining a tripartite graph T_i which can be built upon to obtain a realization of d_{RR} with independence number 3, as before.

We have that $d_{RR} = k^a(k-1)^{b+2k}$ and that the condition on $a+b$ implies that $R(d_{RR}) = 3$ and $k \geq 3$. We will write the length of d_{RR} as $a + b + 2k = 3l + i$ for some $l \in \mathbb{N}$, $0 \leq i < 3$. Note $l \geq 2$.

	$i = 0$	$i = 1$	$i = 2$
d	$(k-l+1)^a(k-l)^{l-a}$	$(k-l)^a(k-l-1)^{l+1-a}$	$(k-l+1)^a(k-l)^{l-a}$
$e = f$	$(k-l)^l$	$(k-l)^l$	$(k-l-1)^{l+1}$
D	$l(k-l)+a$	$(l+1)(k-l)+a-l-1$	$l(k-l)+a$
$E = F$	$l(k-l)$	$l(k-l)$	$(l+1)(k-l-1)$
$F' = E'$	$\frac{1}{2}D$	$\frac{1}{2}D$	$\frac{1}{2}D$
$D' = E - \frac{1}{2}D$	$\frac{1}{2}[l(k-l)-a]$	$\frac{1}{2}[(l-1)(k-l)+l-a+1]$	$\frac{1}{2}[(l+2)(k-l)-(2l-2+a)]$
caveats			$k \geq l+3$

One checks that the following relationships hold:

$$\frac{3}{2}l > k - 1 \geq l > a.$$

Consider the caveat in the case $i = 2$. Since $k \geq l + 1$, if $k \leq l + 2$ we have that either $k = l + 1$ or $k = l + 2$. If $k = l + 1$ then $a + b + 2k = 3l + 2$ implies that $a + b = k - 1$, contrary to hypothesis. If $k = l + 2$ then $a + b + 2k = 3l + 2$ implies that $a + b = k - 4$. The length of d_R is $a + b + k = 2k - 4$ then, which is even. By Theorem 2.10, $\alpha(d_{RR}) = R(d_{RR})$. We can therefore assume the caveat holds.

We leave the rest of the details to the reader. □

The preceding lemmas yield the following theorem:

Theorem 3.8. *If $m \geq 2$ then $\alpha(d) = R(d)$ provided $0 < a + b < k$. If $n \geq 2$ then $\alpha(d) = R(d)$ provided $0 < a + b < k - 1$.*

Proof. Apply the method of proof as was used in Theorem 2.10, using Lemmas 3.6 and 3.7. □

4. Bipartite Realizations of Semi-regular Degree Sequences

In Section 2 we presented a number of constructions for graphs with independence number 2. Notice that the complements of such graphs are triangle-free graphs. There are many results regarding the structure of triangle-free graphs. In [2] Andrásfai, Erdös, and Sós prove what is perhaps the most famous of these theorems:

Theorem 4.1. *If G is a triangle-free graph on n vertices and $\delta > \frac{2}{5}n$ then G is bipartite.*

If G is both semi-regular and bipartite, more can be said about the structure of G.

Fact 4.2. *Let $D = r^A(r-1)^B$ be graphic. Let $N = A+B$ be odd. If G is a bipartite realization of D and $\delta > \frac{1}{3}N$ then $V(G)$ has partition sizes $\lceil \frac{N}{2} \rceil$ and $\lfloor \frac{N}{2} \rfloor$.*

Proof. First notice that if either A or B is zero then G is regular. Then since G is bipartite, G must have partitions of equal sizes so that N is even, contrary to hypothesis. Thus we can assume that neither A nor B is zero. In particular, $\delta = r - 1$.

Suppose $V(G) = X \cup Y$. Assume $|Y| > |X|$. Counting edges of G in two ways we obtain $|X|r \geq |Y|(r - 1)$ so that $|Y| \geq (|Y| - |X|)r$. Suppose $|Y| - |X| \geq 2$. Then

$$|Y| \geq 2r = 2(r - 1) + 2 = 2\delta + 2 > \frac{2}{3}N + 2.$$

But then

$$|X| = N - |Y| < N - (\frac{2}{3}N + 2) = \frac{1}{3}N - 2 < \frac{1}{3}N < \delta.$$

This can not be, since for any $y \in Y, \Gamma(y) \subseteq X$ so that $|X| \geq |\Gamma(y)| \geq \delta$. Thus $|Y|-|X| < 2$. Since $|Y|-|X| > 0$ we must have that $|Y|-|X| = 1$. Thus $|Y| = \lceil \frac{N}{2} \rceil$ and $|X| = \lfloor \frac{N}{2} \rfloor$ as claimed. □

Theorem 4.3. *Let $D = r^A(r - 1)^B$ be graphic. Let $N = A + B$. If N is odd, $r - 1 > \frac{1}{3}N$ and $A \geq N - r$, then D has a bipartite realization if and only if $A = N - r$ and $r \leq \lceil \frac{N}{2} \rceil$.*

Proof. Let $l = \left\lfloor \frac{N}{2} \right\rfloor$ so that $N = 2l + 1$. Suppose G is a bipartite realization of d with vertex classes X and Y. By Fact 4.2 we have $|X| = l$ and $|Y| = l + 1$. Then $r \leq l + 1 = \left\lceil \frac{N}{2} \right\rceil$. Let x be the number of vertices of degree r in X and count the edges of G in two ways to obtain

$$rx + (r-1)(l-x) = r(A-x) + (r-1)((l+1) - (A-x)).$$

Solving for A we obtain

$$A = 2x + 1 - r \leq 2l + 1 - r = N - r$$

since $x \leq |X| = l$. Now $A \geq N - r$ by assumption so that $A = N - r$.

Now suppose $A = N - r$, $r \leq \left\lceil \frac{N}{2} \right\rceil$ and that D satisfies the given hypotheses. We construct a bipartite graph which realizes $D = r^{N-r}(r-1)^r$ by defining $V(G) = X \cup Y$ where

$$X = \{x_1, x_2, ..., x_l\}$$
$$Y = \{y_1, y_2, ..., y_{l+1}\}$$

and the edges of G consist of r disjoint semi-perfect matchings between Y and X. Since each of the r vertices in Y left unmatched by a matching has degree $r - 1$, while the remaining $l + 1 - r$ have degree r, and every vertex in X has degree r, $d(G) = r^{l+1-r+l}(r-1)^r = r^{N-r}(r-1)^r$ as claimed. □

Theorem 4.4. *Let $D = r^A(r-1)^B$ be graphic. Let $N = A + B$. If N is odd, $r - 1 > \frac{1}{3}N$ and $B \geq N - r + 1$, then D has a bipartite realization if and only if $B = N - r + 1$ and $r \leq \left\lceil \frac{N}{2} \right\rceil$.*

Proof. Similar to that of Theorem 4.3. □

Theorems 4.3 and 4.4 are used extensively in Section 5. The following lemmas are used to prove Theorems 4.6 and 4.7, the results of which complement those of Theorem 3.8.

Lemma 4.5. *If G is any graph containing m vertices of degree r then G has an independent set of size at least $\left\lceil \frac{m}{r+1} \right\rceil$ which consists of vertices of degree r.*

Proof. Let

$$S_0 = \{v \in V(G) | d(v) = r\}$$

and choose $x_0 \in S_0$ arbitrarily. Then, for each $i > 0$ such that $m - i(r+1) > 0$, define

$$S_i = S_{i-1} \backslash (\{x_{i-1}\} \cup \Gamma(x_{i-1}))$$

where x_i is chosen arbitrarily in S_i. Then the set $S = \{x_0, x_1, \ldots x_{\lceil \frac{m}{r+1} \rceil - 1}\} \subseteq S_0$ is independent in G. □

Theorem 4.6. *Suppose $d = k^{m(k+1)+a}(k-1)^b$ is graphic with $a + b = k$, where $a \leq k$ and $b \leq k - 1$. Then $\alpha(d) = R(d)$ if $k \leq 2$ and $\alpha(d) = R(d) + 1$ if $k > 2$.*

Proof. Note that $m \geq 1$ and $k \geq 2$, else d is not graphic. If $k = 2$ then $b = 0$ and $a = 2$ since $ka + (k-1)b$ is even, and thus $d = 2^{3m+2}$. The graph G which consists of $m - 1$ triangles together with a copy of C_5 is an optimal realization of d.

Assume now that $k > 2$ but that $\alpha(d) \neq R(d) + 1$ for some fixed d satisfying the hypotheses of the theorem. Then $\alpha(d) = R(d) = m + 1$. Thus there exists an optimal realization G of d. By Lemma 4.5 G contains an independent set

$S = \{x_1, x_2, \ldots, x_{m+1}\}$ of $m+1$ vertices all of degree k. Since $\alpha(G) = m+1$, S is therefore maximally independent in G. Hence,

$$\left|\bigcup_{i=1}^{m+1} \Gamma(x_i)\right| = |V(G)\backslash S| = (m+1)k - 1.$$

Thus there exists a unique pair of indices p and q, $p \neq q$, such that

$$|\Gamma(x_p) \cap \Gamma(x_q)| = 1$$

while for all other pairs i and j with $i \neq j$ we have

$$|\Gamma(x_i) \cap \Gamma(x_j)| = 0.$$

Without loss of generality assume $p = 1$ and $q = 2$. Observe that, for $i \geq 3$, the subgraph $G[\{x_i\} \cup \Gamma(x_i)]$ of G is a clique: If $a, b \in \Gamma(x_i)$ and $ab \notin E(G)$ then the set $S\backslash\{x_i\} \cup \{a, b\}$ is an independent set of size $m + 2 > \alpha(G)$. Thus,

$$G \cong G' \cup \bigcup_{i=3}^{m+1} K_{k+1}$$

where $G' = G[\{x_1, x_2\} \cup \Gamma(\{x_1, x_2\})]$.

Notice that $\alpha(G) = \alpha(G') + m - 1$ so that $\alpha(G') = 2$. Now, the $m-1$ copies of K_{k+1} account for $(k+1)(m-1)$ vertices of degree k in G so that $d(G') = k^{k+1+a}(k-1)^b$. Since there are no edges in G between G' and the remaining vertices the complementary graph $\overline{G'}$ is triangle-free with degree sequence $d(\overline{G'}) = (a+b+1)^b(a+b)^{k+1+a}$. Thus

$$\delta_{\overline{G'}} = a + b = k > \frac{2}{5}(2k+1)$$

since $k > 2$. By Theorem 4.1, then, $\overline{G'}$ is bipartite. Note that the hyposthese of Theorem 4.4 are suatisfied, and hence it must be that $a = 0$ and $b = k$. This is a contradiction since $b < k$ by assumption. Hence no such d exists and $\alpha(d) = R(d)+1$ under the assumptions of the theorem. □

The analogous result for $n \geq 1$ has a similar, although not identical, proof.

Theorem 4.7. *Let $d = k^a(k-1)^{b+nk}$ be graphic with $a + b = k - 1$.*

(1) *If $b = 0$, then $\alpha(d) = R(d)$.*
(2) *If $b > 0$ and $k \leq 4$, then $\alpha(d) = R(d)$.*
(3) *If $b > 0$, $k = 5$, and $a > 0$, then $\alpha(d) = R(d)$.*
(4) *If $b > 0$, $k = 5$, and $a = 0$, then $\alpha(d) = R(d) + 1$.*
(5) *If $b > 0$ and $k > 5$, then $\alpha(d) = R(d) + 1$.*

Proof. (1) If $b = 0$, then the graph which consists of n copies of K_k and one copy of K_{k-1}, together with edges that match the vertices of K_{k-1} into the vertices of one copy of K_k is an optimal realization of $d = k^{k-1}(k-1)^{nk}$.

(2) Suppose $b > 0$. If $k \in \{1, 2\}$ then d is not graphic. If $k = 3$ then $d = 3^0 2^{2+3n} = 2^{2+3n}$ and the graph consisting of $n-1$ disjoint triangles and C_5 is an optimal realization of d. If $k = 4$ then $d = 4^1 3^{2+4n}$ so that $d_R = 4^1 3^6$. An optimal realization of d_R is shown in Figure 4 so that $\alpha(d) = R(d)$ in this case as well.

(3) If $b > 0$, $k = 5$, and $a > 0$, then $d = 5^2 4^{2+5n}$ so that $d_R = 5^2 4^7$. An optimal realization of d_R is shown in Figure 5 so that $\alpha(d) = R(d)$.

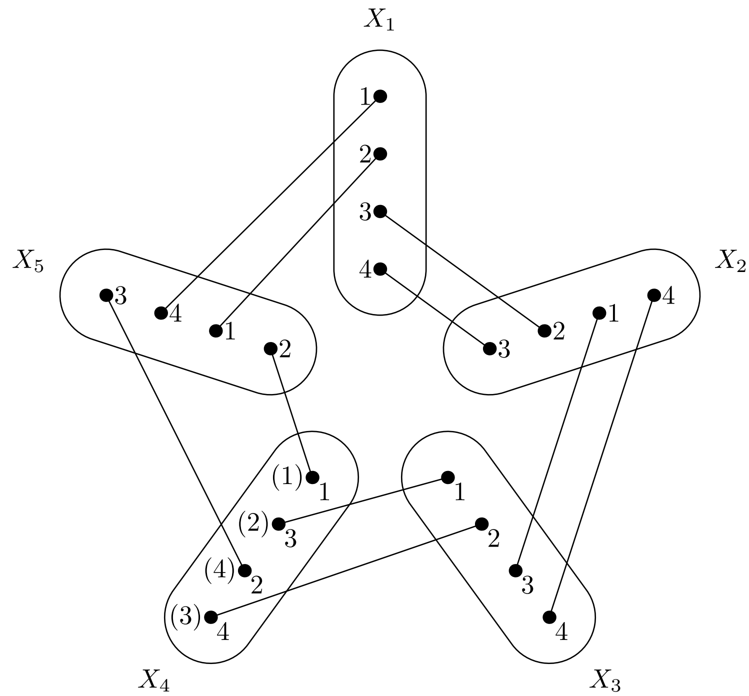


FIGURE 4. A realization of the sequence $d_R = 4^1 3^6$ which has independence number 2.

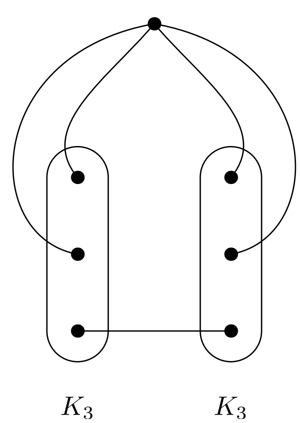


FIGURE 5. A realization of the sequence $d_R = 5^2 4^7$ which has independence number 2.

(4) If $k = 5$ and $a = 0$, then $d = 5^0 4^{4+5n} = 4^{5n+4} 3^0$ so that $\alpha(d) = R(d) + 1$ by Theorem 4.6.

(5) Suppose that $b > 0$ and $k > 5$. If $\alpha(d) \neq R(d)+1$ for some such sequence d, then $\alpha(d) = R(d)$. So there exists a realization G of d with $\alpha(G) = R(d) = n+1$. Then the set $S = \{x_1, x_2, \ldots, x_{n+1}\}$ of $n+1$ vertices of degree $k-1$,

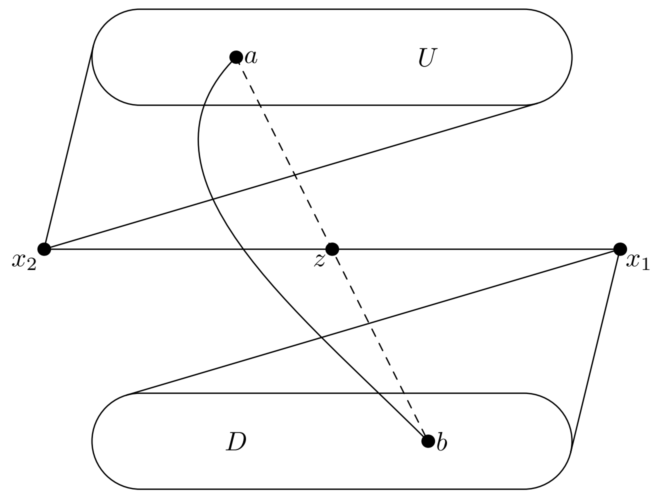


FIGURE 6. The structure of the subgraph P.

guaranteed to exist by Lemma 4.5, is maximally independent in G. Hence,

$$\left|\bigcup_{i=1}^{n+1}\Gamma(x_i)\right| = |V(G)\backslash\{x_1, x_2, \ldots, x_{n+1}\}| = (n+1)(k-1)-1.$$

Thus there exists a unique pair of indices p and q, $p \neq q$, such that

$$|\Gamma(x_p) \cap \Gamma(x_q)| = 1$$

while for all other pairs i and j, $i \neq j$, we have

$$|\Gamma(x_i) \cap \Gamma(x_j)| = 0.$$

Without loss of generality assume $p = 1$ and $q = 2$. Observe then that for $i \geq 3$ the subgraph induced by $\{x_i\} \cup \Gamma(x_i)$ is a clique of size k: If $a, b \in \Gamma(x_i)$ with $ab \notin E(G)$ then we can replace x_i in the set S by a, b to obtain an independent set of size $n+2 > \alpha(G)$.

Let $P = G[\{x_1, x_2\} \cup \Gamma(\{x_1, x_2\})]$. We claim that $\alpha(P) = 2$. For if $I \subseteq V(P)$ is independent in P and $|I| > 2$ then the set $I \cup \{x_3, \ldots, x_{n+1}\}$ is independent in G and $|I \cup \{x_3, \ldots, x_{n+1}\}| > n+1 = \alpha(G)$. We need to examine the structure of P in more detail. A diagram is shown in Figure 6.

Let $\{z\} = \Gamma(x_1) \cap \Gamma(x_2)$. Let $U = \Gamma(x_2)\backslash\{z\}$ and $D = \Gamma(x_1)\backslash\{z\}$. Then the subgraph of P induced by $\{x_1\} \cup D$ is a clique of size $k-1$, otherwise x_1 and a pair of non-adjacent vertices in D would form an independent set of size 3 in P. Similarly, $\{x_2\} \cup U$ is a clique of size $k-1$.

Now, since $d_G(z) \leq k$ we have that $d_P(z) \leq k$. Since two neighbors of z are x_1, x_2, we need to account for at most $k-2$ other neighbors of z in P.

Notice that if $U \subseteq \Gamma(z)$, then $d_P(z) = k$ and $D \cap \Gamma(z) = \emptyset$. This means that, in G, every vertex of D is joined to a vertex of $\Gamma(x_i)$ for some $i \geq 3$. Hence there are at least $|D| + 1 = k-1$ vertices of degree k in G, contrary to assumption. Similarly we must not have that $D \subseteq \Gamma(z)$.

So let $U_1 = \Gamma(z) \cap U$ and $U_2 = U \backslash U_1$, and let $D_1 = \Gamma(z) \cap D$ and $D_2 = D \backslash D_1$. Note that

$$\{ud | u \in U_2, d \in D_2\} \subseteq E(P),$$

otherwise $\{u, d, z\}$ is independent in P for some $u \in U_2$ and some $d \in D_2$. Also, since both $D \cup \{x_1\}$ and $U \cup \{x_2\}$ are cliques of size $k-1$ and $\Delta_G \leq k$, we have that $|U_2| \leq 2$ and $|D_2| \leq 2$. Hence

$$k \geq d(z) \geq 2(k-4) + 2 = 2k - 6.$$

Hence $k \leq 6$. Since $k > 5$ by assumption, we must have that $k = 6$.

But now if $k = 6$, then the conditions $d = 6^a 5^{b+6n}$ with $b > 0$ and $a + b = 5$ imply that either $a = 3$ and $b = 2$, or $a = 1$ and $b = 4$. Here we have that $|D| = |U| = 4$ and that $|D_2| \leq 2$ and $|U_2| \leq 2$. Thus $|\Gamma(z) \cap (D \cup U)| \geq 4$ so that $d(z) = 6$. Thus $|D_2| = |U_2| = 2$. Now, both elements of D_2 are joined to both elements of U_2 so that each of these 4 vertices has degree at least 6. But this means that G contains at least 5 vertices of degree 6 contradicting the fact that a is either 3 or 1.

□

5. Graphs Which are Homomorphic to C_5

We are now left only to consider the existence of optimal realizations for those semi-regular graphic sequences which are left or right minimal and of odd length. Recall that d is left minimal means that $d = d_L = k^{k+1+a}(k-1)^b$ and d_C is not graphic, while d is right minimal means that $d = d_R = k^a(k-1)^{b+nk}$ and d_C is not graphic, where $a < k+1$ and $b < k$. Since optimal realizations of these sequences have independence number 2, it is enough to find triangle-free realizations of the complements of such sequences, or to show such a graph cannot exist.

The 5-cycle, C_5, is triangle-free, as is any graph which is homomorphic to it. The following theorem, an extension of Theorem 4.1, gives a sufficient condition for recognizing a triangle-free graph as one which is homomorphic to C_5:

Theorem 5.1. *Suppose G is a triangle-free graph on n vertices. If $\delta > \frac{3}{8}n$ then G is homomorphic to C_5. If $\delta > \frac{2}{5}n$ then G is bipartite.*

Proof. See [11] and [2]. □

For further results regarding the structure of triangle-free graphs where δ approaches $\frac{1}{3}$ from above, see [3], [4] and [11].

So suppose $D = r^A(r-1)^B$ is a graphic semi-regular sequence of odd length $N = A + B$. Write $N = 5l + j$ for some $l \in \mathbb{N}$, some $j \in \{0, 1, 2, 3, 4\}$. The value of Theorem 5.1 to us is illustrated by the following corollaries:

Corollary 5.2. *If G is a triangle-free realization of $D = r^A(r-1)^B$, where $r > 2l+2$ and $A + B = 5l + j$ for some $j \in \{0, 1, 2, 3, 4\}$, then G is bipartite.*

Corollary 5.3. *If G is a triangle-free realization of $D = (2l+2)^A(2l+1)^B$ where $N = A + B = 5l + j$ for some $j \in \{0, 1, 2, 3, 4\}$, then G is bipartite for $j = 0, 1,$ or 2 and G is homomorphic to C_5 for $j = 3, 4$ provided $l > 3j - 8$.*

Corollary 5.4. *If G is a triangle-free realization of $D = (2l+1)^A 2l^B$ where $A + B = 5l + j$ for some $j \in \{0, 1, 2, 3, 4\}$, then G is homomorphic to C_5 provided $l > 3j$.*

It is easy to see that if G is homomorphic to C_5 but not surjectively so then G is bipartite. Theorems 4.3 and 4.4 give necessary and sufficient conditions for determining existence of bipartite graphs under certain conditions. Notice that $\overline{d_L} = (a+b+1)^b(a+b)^{a+k+1}$ and $\overline{d_R} = (a+b)^{b+k}(a+b-1)^a$ so that these theorems will prove sufficient for our purposes. In particular, for $j = 0$, the above corollaries imply that if a triangle-free realization of $D = r^A(r-1)^B$, for $r \geq 2l+1$, exists, then such a graph either maps homomorphically onto C_5 or is bipartite.

If G maps homomorphically onto C_5, then there is a surjective graph homomorphism $\phi : V(G) \to V(C_5)$. Assuming the vertices $\{x_1, x_2, x_3, x_4, x_5\}$ of C_5 are arranged cyclically, let $X_i = \phi^{-1}(x_i)$ for each i. Then if $n_i = |X_i|$, $n_i > 0$ for each i, and $\sum_{i=1}^5 n_i = |V(G)|$. With this notation we have the following:

Theorem 5.5. *Let G be a realization of $D = r^A(r-1)^B$ which maps homomorphically onto C_5, where $r \geq 2l+1$, $N = A+B = 5l+j$, $j \in \{0,1,2,3,4\}$. Let P denote the number of vertex classes which contain a vertex of degree r. Then*

(1) $P \leq 2N - 5(r-1)$.
(2) $3(r-1) - N \leq n_i \leq N - 2(r-1)\ \forall\ i \in [5]$

Proof. Let $\delta_i = 1$ if X_i contains a vertex of degree r; otherwise $\delta_i = 0$. We must have that:

$$\begin{aligned} n_1 + n_3 &\geq r - 1 + \delta_2 \\ n_2 + n_4 &\geq r - 1 + \delta_3 \\ n_3 + n_5 &\geq r - 1 + \delta_4 \\ n_4 + n_1 &\geq r - 1 + \delta_5 \\ n_5 + n_2 &\geq r - 1 + \delta_1 \end{aligned}$$

Summing, $2\sum n_i \geq 5(r-1) + \sum \delta_i$. Since $P = \sum \delta_i$, this proves part 1.

For part 2, notice that $\sum_{i=2}^5 n_i \geq 2(r-1)$ so that $n_1 \leq N - 2(r-1)$. By symmetry of the inequalities above, this upper bound holds for each $i \in [5]$. Then, for example, $n_1 + n_3 \geq (r-1)$ now yields $n_3 \geq 3(r-1) - N$ and this lower bound holds for each $i \in [5]$ by symmetry. ☐

Corollary 5.6. *Let G be a realization of $D = (2l+1)^A 2l^B$ which maps homomorphically onto C_5, where $N = A+B = 5l+j$, $j \in \{0,1,2,3,4\}$. Let P denote the number of vertex classes which contain a vertex of degree $2l+1$. Then*

(1) $P \leq 2j$
(2) $l - j \leq n_i \leq l + j$ *for each* $i \in [5]$

Thus, for $j = 0$ and $r \geq 2l+1$, any triangle-free realization of $D = r^A(r-1)^B$ with $A > 0$ is bipartite.

In [16] we give templates for constructing triangle-free realizations of any graphic sequence $D = r^A(r-1)^B$ with $r \leq 2l$, where $N = A+B = 5l+j$, $j \in \{0,1,2,3,4\}$ is of odd length. We also give an in depth treatment of those cases which arise from the finite number of values of l not covered by Corollaries 5.4 and 5.3. The details are tedious, so we present here only the case for $j = 0$ in hopes of conveying the spirit of our approach. A summary of results is given at the end of this section.

5.1. **A Method for Constructing Triangle-free Semi-regular Graphs.** The basic idea is to construct a graph which is homomorphic to C_5 which is $2t$-regular for some $t \in \mathbb{N}$ and whose edge set either contains a fixed matching J or is disjoint from J. We then obtain a triangle-free semi-regular graph with either $\Delta = 2t$ and $\delta = 2t - 1$ by removing edges of J, or a triangle-free semi-regular graph with $\Delta = 2t + 1$ and $\delta = 2t$ by adding edges of J. For clarity, we will assume that the number of vertices, N, is divisible by 5 and write $N = 5l$ for some $l \in \mathbb{N}$. (See [16] for variations on the construction when $N \not\equiv 0 \pmod 5$.)

To begin, divide the vertices into 5 classes, each of size l. We think of these classes as being arranged cyclically and refer to them as X_1, X_2, X_3, X_4, X_5 so that the homomorphism with C_5 is clear. Determine the fixed matching J by matching $\lfloor \frac{l}{2} \rfloor$ vertices in X_1 with $\lfloor \frac{l}{2} \rfloor$ vertices in X_2, and $\lfloor \frac{l}{2} \rfloor$ other vertices in X_1 with $\lfloor \frac{l}{2} \rfloor$ vertices in X_5. Match the remaining vertices of X_2 and X_5 with vertices in X_3 and X_4, respectively. Now there are an equal number of unmatched vertices in X_3 and X_4 so that these can be matched with each other. If l is even, J is a perfect matching while, if l is odd, J is a semi-perfect matching which leaves a lone vertex in X_1 unmatched.

We use the fixed matching J and the following corollary to Proposition 2.4 to label our vertices in a useful way:

Corollary 5.7. *Suppose n edges match a set $X = \{x_1, x_2, \ldots, x_n\}$ of labeled vertices into a set Y of n unlabeled vertices. Then the elements of Y can be labeled by $\{y_1, y_2, \ldots, y_n\}$ in such a way that, relative to this labeling, the given edges are precisely $P_1[X, Y]$. Similarly, there is a labeling of Y such that relative to this labeling the given edges are precisely $P_1[Y, X]$.*

Proof. See Proposition 2.4 for explanation of notation and Figure 7 for an example of that notation. □

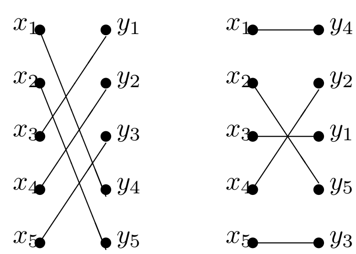


FIGURE 7. Both graphs have edge set $P_2[X, Y] = P_3[Y, X]$ relative to the labelings shown.

Now label the vertices of X_1 with the labels $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1\}$ randomly. By Corollary 5.7, the vertices of X_2 and X_5 can be labeled with the labels $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\}$, and $\{x_1^5, x_2^5, x_3^5, x_4^5, x_5^5\}$ respectively, in such a way that the edges of J which match X_1 into X_2 are contained in $P_1[X_1, X_2]$ and the edges of J which match X_1 into X_5 are contained in $P_1[X_1, X_5]$. These labelings of X_2 and X_5 similarly determine (non-uniquely) labelings of X_3 and X_4 with the respective label sets $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3\}$ and $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4\}$ so that the edges of J which match X_2 into X_3 are contained in $P_1[X_2, X_3]$ and the edges of J which match X_5 into X_4 are contained in $P_1[X_5, X_4]$. Now X_3 and X_4 have been labeled independently of each other. By assigning to X_4 *a second, possibly different, labeling* with the label set $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4\}$, we can arrange that the edges of J matching vertices of X_4

into X_3 are contained in $P_1[X_4, X_3]$ *relative to the new labeling of* X_4. An example of such a labeling, together with a possible choice of edges J, is shown for $N = 25$ in Figure 8.

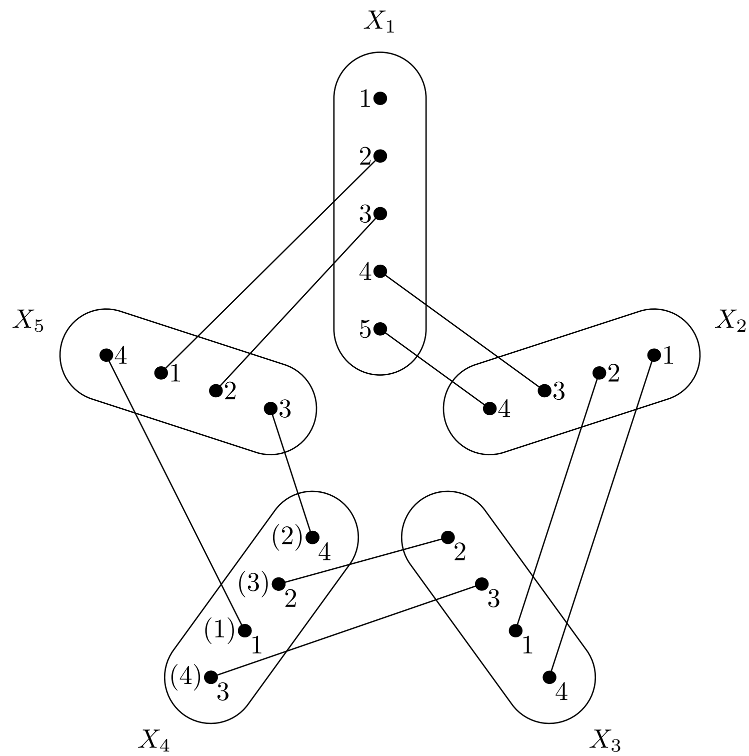
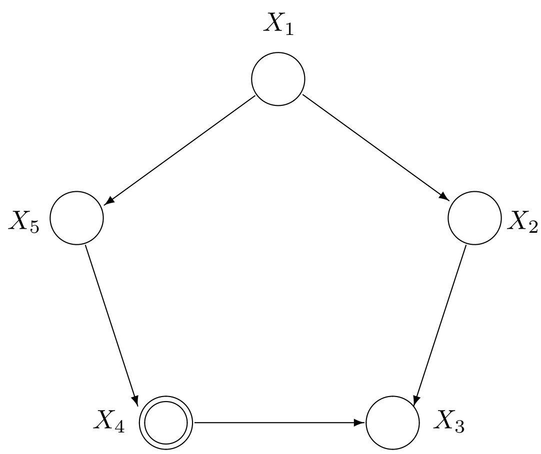
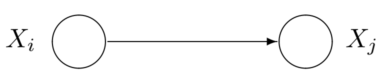


FIGURE 8. Edges represent the elements of J. Labels shown are subscripts of labels used in each X_i. A *relabeling* of X_4 is shown in parenthesis.

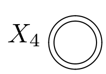
The diagram below represents a graph whose labels are begun in X_1, with a relabeling in X_4. It has two possible interpretations: either the graph contains the edges of J or its edge set is disjoint from J.



Fix $\delta \in \{0,1\}$. If $\delta = 0$, let t be any fixed integer $0 < t \le l$. If $\delta = 1$, let t be any fixed integer $0 \le t < l$. In either case $t + \delta \le l$ so that arrows between consecutive vertex classes represent the edges of t perfect matchings so that the graph is $2t$-regular. More precisely, edges between consecutive vertex classes are represented as shown schematically below. If $\delta = 0$, we call the graph G^-. We have that G^- is triangle-free and $2t$-regular for $0 < t \le l$. Moreover, $E(G^-) \cap J = J$ so by removing a suitable number of edges of J from $E(G^-)$ we obtain a triangle-free realization for any graphic sequence $D = (2t)A(2t-1)^B$ where $A + B = 5l$ and $t \le l$.



$[X_i, X_j] = \cup_{q=1+\delta}^{t+\delta} P_q[X_i, X_j]$



a relabeling may occur in vertex class X_4; matchings between X_4 and X_3 are to be interpreted relative to the *relabeling* of X_4

If $\delta = 1$, we call the graph constructed G^+. Note G^+ is triangle-free, $2t$-regular for $0 \leq t < l$, and $E(G^+) \cap J = \emptyset$. We can thus add a suitable number of edges of J to $E(G^+)$ in order to obtain a triangle-free realization of any graphic sequence $D = (2t+1)^A (2t)^B$ where $A + B = 5l$ and $t < l$.

5.2. **Summary of Results.** If d is right minimal and of odd length, then $d = d_R = k^a(k-1)^{b+k}$ where $a < k+1$, $b < k$, and d_C is not graphic. Let $N = a+b+k$. In the cases where N is not divisible by 5, there exist constructions similar to the one described above which show that $\alpha(d) = R(d)$ for each such d satisfying $a + b \leq 2\left\lfloor \frac{N}{5} \right\rfloor$. As the value of $a+b$ increases relative to the length of the sequence, optimal realizations become more rare. Table 1 gives necessary and sufficient conditions for determining when $\alpha(d) = R(d)$ and so summarizes under which conditions optimal realizations of d exist.

	$a+b = 2\left\lfloor \frac{N}{5} \right\rfloor + 1$	$a+b = 2\left\lfloor \frac{N}{5} \right\rfloor + 2$	$a+b \geq 2\left\lfloor \frac{N}{5} \right\rfloor + 3$
$N \equiv 0 \pmod 5$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$
$N \equiv 1 \pmod 5$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$
$N \equiv 2 \pmod 5$	$\alpha(d) = R(d) \iff a \geq \left\lfloor \frac{N}{5} \right\rfloor$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$
$N \equiv 3 \pmod 5$	$\alpha(d) = R(d) \iff a+1 \geq \left\lfloor \frac{N}{5} \right\rfloor$	$\alpha(d) = R(d) \iff b = 0$	$\alpha(d) = R(d) \iff b = 0$
$N \equiv 4 \pmod 5$	$\alpha(d) = R(d) \iff a+2 \geq \left\lfloor \frac{N}{5} \right\rfloor$	$\alpha(d) = R(d) \iff a \geq 2\left\lfloor \frac{N}{5} \right\rfloor$	$\alpha(d) = R(d) \iff b = 0$

TABLE 1. Necessary and sufficient conditions needed to ensure that $\alpha(d) = R(d)$ where $d = k^a(k-1)^{b+k}$ is right minimal of odd length. Note $\alpha(d) = R(d)$ for all $a + b \leq 2\left\lfloor \frac{N}{5} \right\rfloor$.

Similarly, if d is left minimal and of odd length, then $d = d_L = k^{k+1+a}(k-1)^b$ where $a < k+1$, $b < k$ and d_C is not graphic. Letting $N = k+1+a+b$, we note here that constructions of optimal realizations exist to show that $\alpha(d) = R(d)$ provided $a + b < 2\left\lfloor \frac{N}{5} \right\rfloor$ or $a + b < 2\left\lfloor \frac{N}{5} \right\rfloor$ and $b = 0$. Table 2 summarizes necessary and sufficient conditions for determining if $\alpha(d) = R(d)$ holds for all other values of $a + b$.

	$a+b=2\left\lfloor\frac{N}{5}\right\rfloor, b>0$	$a+b=2\left\lfloor\frac{N}{5}\right\rfloor+1$	$a+b\geq 2\left\lfloor\frac{N}{5}\right\rfloor+2$
$N\equiv 0 \pmod 5$	$\alpha(d)=R(d)\iff a=0$	$\alpha(d)=R(d)\iff a=0$	$\alpha(d)=R(d)\iff a=0$
$N\equiv 1 \pmod 5$	$\alpha(d)=R(d)$	$\alpha(d)=R(d)\iff a=0$	$\alpha(d)=R(d)\iff a=0$
$N\equiv 2 \pmod 5$	$\alpha(d)=R(d)$	$\alpha(d)=R(d)\iff a=0$	$\alpha(d)=R(d)\iff a=0$
$N\equiv 3 \pmod 5$	$\alpha(d)=R(d)$	$\alpha(d)=R(d)\iff$ either $b=\left\lfloor\frac{N}{5}\right\rfloor+1$ or $a=0$	$\alpha(d)=R(d)\iff a=0$
$N\equiv 4 \pmod 5$	$\alpha(d)=R(d)$	$\alpha(d)=R(d)\iff b\geq\left\lfloor\frac{N}{5}\right\rfloor$	$\alpha(d)=R(d)\iff a=0$

Table 2. Necessary and sufficient conditions needed to ensure that $\alpha(d)=R(d)$ where $d=k^{k+1+a}(k-1)^b$ is left minimal of odd length. Note $\alpha(d)=R(d)$ for all $a+b<2\left\lfloor\frac{N}{5}\right\rfloor$ and for $a+b=2\left\lfloor\frac{N}{5}\right\rfloor$ with $b=0$.

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