Correction of Theorem 4.13

Theorem 4.13 (Discrete Floquet's Theorem) If $\Phi(t)$ is a fundamental matrix for the Floquet system (4.16), then $\Phi(t+p)$ is also a fundamental matrix and $\Phi(t+p) = \Phi(t)C$, where

$$C = \Phi^{-1}(0)\Phi(p). \tag{4.21}$$

Furthermore, there is a nonsingular matrix function $P(t)$ and a nonsingular constant matrix B such that

$$\Phi(t) = P(t)B^t, \tag{4.22}$$

where $P(t)$ is periodic with period p.

Proof. Assume $\Phi(t)$ is a fundamental matrix for the Floquet system (4.16). If $\Psi(t) \equiv \Phi(t+p)$, then $\Psi(t)$ is nonsingular for all t, and

$$\begin{aligned}\Psi(t+1) &= \Phi(t+p+1) \\ &= A(t+p)\Phi(t+p) \\ &= A(t)\Psi(t),\end{aligned}$$

so $\Phi(t+p)$ is a fundamental matrix for Eq. (4.16). By Theorem 4.10, there is a nonsingular matrix C such that

$$\Phi(t+p) = \Phi(t)C,$$

for all integers t. Letting $t = 0$ and solving for C, we get that Eq (4.21) holds. By Lemma 4.1, there is a nonsingular matrix B so that $B^p = C$. Let

$$P(t) \equiv \Phi(t)B^{-t}. \tag{4.23}$$

Note that $P(t)$ is nonsingular for all t and since

$$\begin{aligned}P(t+p) &= \Phi(t+p)B^{-(t+p)} \\ &= \Phi(t)CB^{-p}B^{-t} \\ &= \Phi(t)B^{-t} \\ &= P(t),\end{aligned}$$

$P(t)$ has period p. Solving Eq. (4.23) for $\Phi(t)$, we get Eq. (4.22).

Definition 4.2. Let $\Phi(t)$ and C be as in Floquet's theorem (Theorem 4.13), then the eigenvalues μ of the matrix C are called the "Floquet multipliers" of the Floquet system (4.16).

Since fundamental matrices of a linear system are not unique, we must show that Floquet multipliers are well defined. Let $\Phi(t)$ and $\Psi(t)$ be fundamental matices for the Floquet system (4.16) and let

$$C_1 = \Phi^{-1}(0)\Phi(p) \quad \text{and} \quad C_2 = \Psi^{-1}(0)\Psi(p).$$

It remains to show that C_1 and C_2 have the same eigenvalues. By Theorem 4.10 there is a matrix F so that

$$\Psi(t) \;= \Phi(t)F,$$

for all integers t. Hence,

$$C_2 = \Psi^{-1}(0)\Psi(p) = [\Phi(0)F]^{-1}[\Phi(p)F] = F^{-1}\Phi^{-1}(0)\Phi(p)F = F^{-1}C_1F.$$

Since

$$\begin{aligned}\det(C_2 - \lambda I) &= \det(F^{-1}C_1F - I) \\ &= \det F^{-1}(C_1 - \lambda I)F \\ &= \det(C_1 - \lambda I),\end{aligned}$$

C_1 and C_2 have the same eigenvalues. Hence Floquet multipliers are well defined.

Remark The Floquet multipliers of the Floquet system (4.16) are the eigenvalues of the matrix

$$D \equiv A(p-1)A(p-2)\cdots A(0).$$

To see this let $\Phi(t)$ be the fundamental matrix of the Floquet system (4.16) satisfying $\Phi(0) = I$. Then the Floquet multipliers are the eigenvalues of

$$D = \Phi^{-1}(0)\Phi(p) = \Phi(p).$$

Iterating the equation

$$\Phi(t+1) = A(t)\Phi(t),$$

we get that

$$\begin{aligned}D = \Phi(p) &= [A(p-1)A(p-2)\cdots A(0)]\Phi(0) \\ &= A(p-1)A(p-2)\cdots A(0),\end{aligned}$$

which is the desired result.

Here are some simple examples of Floquet multipliers.

Example 4.14. For the scalar equation

$$u(t+1) \;=\; (-1)^t u(t),$$

the coefficient function $a(t) = (-1)^t$ has minimum period 2, and $d = a(1)a(0) = -1$, so $\mu = -1$ is the Floquet multiplier.

Example 4.15. Find the Floquet multipliers for the Floquet system

$$u(t+1) \;= \begin{bmatrix} 0 & 1 \\ (-1)^t & 0 \end{bmatrix} u(t).$$

The coefficient matrix $A(t)$ is periodic with minimum period 2, so

$$\begin{aligned} D &= A(1)A(0) \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Consequently, $\mu_1 = 1$, $\mu_2 = -1$ are the Floquet multipliers.